Math 5052
Measure Theory and Functional Analysis II
Homework Assignment 9

Prof. Wickerhauser

Due Friday, March 4, 2016

Please do Exercises 12, 16, 17*, 21, 29*, 33, 40, 43, 50, 54*.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: “textbook” refers to “Real Analysis for Graduate Students,” version 2.1, by Richard F. Bass. Some of these exercises originate from that source.

1. Prove that if X is a non-empty set and $\mathcal{T}_1, \mathcal{T}_2$ are two topologies on X, then $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology on X.

2. Prove that if X is a topological space, G is open in X, and A is dense in X, then $\overline{G} = \overline{G \cap A}$.

3. Let $X = \mathbf{R}^2$ with the usual norm topology and write $x \sim y$ if and only if $x = Ax$ for some matrix

$$A = A(\theta) \overset{\text{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

determined by some $\theta \in \mathbf{R}$. (Such an A is called a *Givens matrix* and corresponds to a rotation by θ radians about the origin.)

   a. Show that $\sim$ is an equivalence relation.

   b. Show that the quotient space is homeomorphic to $[0, \infty) \subset \mathbf{R}$ with the usual topology.

4. Prove that every metric space is first countable.

5. Let X be an uncountable set of points and let $\mathcal{T}$ consist of all subsets of A such that A^c is finite and let $\mathcal{T}$ also contain the empty set. Prove that $(X, \mathcal{T})$ is a topological space that is not first countable.

6. Give an example of a metric space which is not second countable.

7. Prove that $A \subset X$ is dense if and only if $A \cap G \neq \emptyset$ for every nonempty open set G.

8. Let $X = \prod_{\alpha \in I} X_\alpha$, where I is a nonempty index set. Prove that a net $\langle x_\beta \rangle$ in X converges to x if and only if the net $\langle \pi_\alpha(x_\beta) \rangle$ converges to $\pi_\alpha(x)$ for every $\alpha \in I$.

9. Let $f : X \to Y$ be a function. Prove that f is continuous if and only if whenever $x \in X$ and $f(x) \in G$ for an open set $G \subset Y$, then there exists an open set $H \subset X$ with $x \in H$ such that $f(H) \subset G$.

10. Let X and Y be topological spaces and fix $y_0 \in Y$. Prove that $X \times \{y_0\}$, with the relative topology derived from $X \times Y$, is homeomorphic to X.

11. Let X, Y, Z be topological spaces, $f : X \to Y$, and $g : Y \to Z$. Prove that if f and g are continuous functions then so is $g \circ f : X \to Z$.

12. Suppose that X, Y are topological spaces and $f : X \to Y$ is a function such that $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X. Is f necessarily continuous? If not, give a counterexample.

13. Prove that $f : X \to Y$ is continuous if and only if the net $\langle f(x_\alpha) \rangle$ converges to $f(x)$ whenever the net $\langle x_\alpha \rangle$ converges to x.

14. Let X be the collection of Lebesgue measurable functions on $[0, 1]$ furnished with the topology of pointwise convergence. Say that $f \sim g$ for $f, g \in X$ if $f = g$ a.e. Describe the quotient topology.

15. A set A has the *Lindelöf property* if every open cover of A has a countable subcover. Prove that a metric space X has the Lindelöf property if and only if X is separable.

16. Find an example of a compact set that is not closed.

17. Show that the product topology on $[0, 1]^{\mathbf{N}}$ and the topology generated by the metric

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|, \qquad x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots),$$

of equation 20.3, section 20.8, textbook p.223, are the same.

18. Let $\{X_\alpha\}$, $\alpha \in I$, be a non-empty collection of topological spaces and let $X = \prod_{\alpha \in I} X_\alpha$. A sequence $\{x_n\}$ in X *converges pointwise* to x if $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each $\alpha \in I$. Prove that x_n converges to x pointwise if and only if x_n converges to x with respect to the product topology of X.

19. Let X be the space of real-valued bounded functions on $[0, 1]$. X can be identified with $\mathbf{R}^{[0,1]}$ furnished with the product topology. Let E be the set of Borel measurable functions on $[0, 1]$.

    a. Show that E is dense in X.

    b. Let N be a subset in $[0, 1]$ that is not Borel measurable and put $f = \chi_N$. Prove that there does not exist a sequence in E that converges to f, but that every neighborhood of f contains points of E.

20. Prove that if I is a nonempty countable set and each X_α, $\alpha \in I$, is second countable, then $\prod_{\alpha \in I} X_\alpha$ is second countable.

21. If X is a metric space, define
$$A^\delta \stackrel{\text{def}}{=} \{x \in X : d(x, A) < \delta\},$$
where $d(x, A) \stackrel{\text{def}}{=} \inf\{d(x, y) : y \in A\}$. For nonempty compact subsets E, F of X, define
$$d_H(E, F) \stackrel{\text{def}}{=} \inf\{\delta > 0 : E \subset F^\delta \text{ and } F \subset E^\delta\}.$$

   a. Prove that d_H is a metric. (This is called the *Hausdorff metric.*)

   b. Suppose X is compact. Is the set of nonempty closed subsets with metric d_H necessarily compact? Supply a proof or a counterexample.

22. Prove that if $\{x_n\}$ is a Cauchy sequence in a metric space X and a subsequence of $\{x_n\}$ converges to a point x, then the full sequence converges to x.

23. Prove that if X is a topological space, then $\mathcal{C}(X)$ is a complete metric space.

24. Prove that a sequence $\{x_n\}$ converges to a point x if and only if every subsequence $\{x_{n_j}\}$ has a further subsequence that converges to x.

25. Let A be a subset of a metric space X. Prove that if A is totally bounded then $\bar{A}$ is also totally bounded.

26. Let X be a topological space in which each singleton $\{x\}$ is closed. Prove that X is a T_1 space.

27. Find two disjoint closed subsets E, F of $\mathbf{R}$ such that $\inf\{|x - y| : x \in E, y \in F\} = 0$.

28. Prove that every metric space is a normal space.

29. Prove that X is a Hausdorff space if and only if every net converges to at most one point.

30. Show that a closed subspace of a normal space is normal.

31. Prove that $[0, 1]^{[0,1]}$ with the product topology is not metrizable.

32. Prove that if X is metrizable and I is countable and nonempty, then X^I is metrizable.

33. Let X be a locally compact Hausdorff space and X^* its one point compactification. A continuous function $f : X \to \mathbf{R}$ is said to *vanish at infinity* if, given $\epsilon > 0$, there exists a compact set $K \subset X$ such that $|f(x)| < \epsilon$ for all $x \in K^c$. Prove that f vanishes at infinity if and only if f is the restriction to X of a continuous function $\bar{f} : X^* \to \mathbf{R}$ with $\bar{f}(\infty) = 0$.

34. Prove that the Alexandroff one-point compactification of $\mathbf{R}^n$ is homeomorphic to the unit n-sphere $\{x \in \mathbf{R}^{n+1} : \|x\| = 1\}$.

35. A sequence $\{f_n\} \subset \mathcal{C}(X)$ is said to *converge uniformly on compact sets* to a function $f \in \mathcal{C}(X)$ if $f_n \to f$ in $\mathcal{C}(K)$ for every compact subset $K \subset X$.

   a. Give an example of a sequence $\{f_n\} \subset \mathcal{C}(\mathbf{R})$ that converges uniformly to 0 on compact sets but that does not converge uniformly to 0 on $\mathbf{R}$.

   b. Let X be a σ-compact locally compact Hausdorff space, $M > 0$, and $\{f_n\}$ an equicontinuous sequence in $\mathcal{C}(X)$ such that $|f_n(x)| \leq M$ for all $x \in X$ and all n. Prove that there exists a subsequence that converges uniformly on compact sets.

36. Show that $\mathbf{R}^{\mathbf{N}}$ is not locally compact.

37. Show that $\mathcal{C}([0,1])$ is not locally compact.

38. Prove that if $\{X_\alpha : \alpha \in I\}$ is a nonempty collection of Hausdorff spaces such that $\prod_{\alpha \in I} X_\alpha$ is locally compact, then each X_α is also locally compact.

39. A real-valued funtion f on a subset $X \subset \mathbf{R}$ is *Hölder continuous of order* α if there exists M such that

$$|f(x) - f(y)| \leq M|x-y|^\alpha$$

for all $x, y \in X$. Suppose $0 < \alpha \leq 1$ and let $X = [0,1]$. Prove that

$$\left\{ f \in \mathcal{C}([0,1]) : \sup_{x \in [0,1]} |f(x)| \leq 1, \sup_{x \neq y \in [0,1]} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq 1 \right\}$$

is compact in $\mathcal{C}([0,1])$.

40. Let $K : [0,1]^2 \to \mathbf{R}$ be continuous and let L be the set of Lebesgue measurable functions f on $[0,1]$ such that $\|f\|_\infty \leq 1$. For $f \in L$, define

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy.$$

Prove that $\{Tf : f \in L\}$ is an equicontinuous family in $\mathcal{C}([0,1])$.

41. Prove that if X is a compact metric space, then $\mathcal{C}(X)$ is separable.

42. Let $X = [0,\infty]$ be the one point compactification of $[0,\infty)$, the nonnegative reals with the usual metric. Let $\mathcal{A}$ be the collection of all finite linear combinations

$$\sum_{j=1}^{n} a_j e^{-\lambda_j x},$$

where the a_j are real and each $\lambda_j \geq 0$.

a. Prove that $\mathcal{A}$ is a dense subset of $\mathcal{C}(X)$.

b. Proe that if f_1 and f_2 are two continuous integrable functions from $[0,\infty)$ to $\mathbf{R}$ that vanish at infinity and which have the same *Laplace transform*, that is,

$$\int_0^\infty e^{-\lambda x} f_1(x)\,dx = \int_0^\infty e^{-\lambda x} f_2(x)\,dx$$

for all $\lambda \geq 0$, then $f_1(x) = f_2(x)$ for all x.

43. Suppose X and Y are compact Hausdorff spaces. Let $\mathcal{A}$ be the collection of real-valued functions in $\mathcal{C}(X \times Y)$ of the form

$$\sum_{i=1}^n a_i g_i(x) h_i(y),$$

where $n \geq 1$, each $a_i \in \mathbf{R}$, each $g_i \in \mathcal{C}(X)$, and each $h_i \in \mathcal{C}(Y)$. Prove that $\mathcal{A}$ is dense in $\mathcal{C}(X \times Y)$.

44. Let X be a compact Hausdorff space and suppose $\mathcal{A}$ is an algebra of continuous functions that separates points. Prove that either $\mathcal{A}$ is dense in $\mathcal{C}(X)$ or else there exists a point $x \in X$ such that $\overline{\mathcal{A}} = \{f \in \mathcal{C}(X) : f(x) = 0\}$.

45. Prove that if $f : [0,1] \to \mathbf{R}$ and $g : [0,1] \to \mathbf{R}$ are continuous functions such that

$$\int_0^1 f(x) x^n\,dx = \int_0^1 g(x) x^n\,dx$$

for $n = 0, 1, 2, \ldots$, then $f = g$.

46. Let X be the closed unit disk in the complex plane. A polynomial in z and $\bar{z}$ is a function of the form

$$P(z) = \sum_{j=0}^n \sum_{k=0}^n a_{jk} z^j \bar{z}^k,$$

where a_{jk} are complex numbers. Prove that if f is a function in $\mathcal{C}(X, \mathbf{C})$, then f can be uniformly approximated by polynomials in z and $\bar{z}$.

47. Prove that if B is a Banach space, then B is connected.

48. Prove that if A is a convex subset of a Banach space, then A is connected.

49. A topological space X is *arcwise connected* (or *path connected*) if for all $x, y \in X$ there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

a. Prove that if X is arcwise connected, then X is connected.

b. Find a counterexample to show that connected does not imply acwise connected.

50. If X is a topological space, a *component* of X is a connected subset of X that is not properly contained in any other connected subset of X. Prove that each $x \in X$ is contained in a unique component of X.

51. A topological space X is *totally disconnected* if its components are all single points.

    a. Prove that the Cantor set with the relative topology derived from the real line is totally disconnected.

    b. Prove that if $\{X_\alpha : \alpha \in I\}$ is a nonempty collection of totally disconnected spaces, then $X \stackrel{\text{def}}{=} \prod_{\alpha \in I} X_\alpha$ is totally disconnected.

52. Prove that a topological space X is connected if and only if for each pair $x, y \in X$ there is a connected subspace of X containing both x and y.

53. Let X be a connected space. Suppose there exists a function $f : X \to \mathbf{R}$ that is continuous and non-constant. Prove that X is uncountable.

54. Suppose $\{A_\alpha : \alpha \in I\}$ is a non-empty collection of connected subsets of a topological space X with the property that $A_\alpha \cap A_\beta \neq \emptyset$ for each $\alpha, \beta \in I$. Prove that $\bigcup_{\alpha \in I} A_\alpha$ is connected.