Math 5052
Measure Theory and Functional Analysis II
Homework Assignment 10

Prof. Wickerhauser

Due Friday, April 1, 2016

Please do Exercises 4, 6, 8, 10*, 13, 14, 18*, 19, 20, 21*

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. Some of these exercises originate from that source.

1. Prove that if A, B, and C are bounded operators from a Hilbert space to itself, then

   a. $A(BC) = (AB)C$;

   b. $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$.

2. Prove that if A is a bounded symmetric operator, then so is A^n for each $n \geq 1$.

3. Suppose $H = \mathbf{C}^n$ and Ax is multiplication of the vector $x \in H$ by an $n \times n$ matrix M. Prove that A^*x is multiplication of x by the conjugate transpose of M.

4. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $F : X \times X \to \mathbf{C}$ a jointly measurable function such that $F(y,x) = \overline{F(x,y)}$ and
$$\iint_{X\times X} |F(x,y)|^2 \, d\mu(x) d\mu(y) < \infty.$$
(This is equation 25.4 on textbook p.353, corrected for complex-valued F.) Define $A : X \to X$ by
$$Af(x) \stackrel{\text{def}}{=} \int_X F(x,y) f(y) \, d\mu(y).$$
(This is equation 25.5 on textbook p.353.) Prove that A is a bounded symmetric operator.

5. If C_1 and C_2 are subsets of a Hilbert space and their closures are compact, prove that the closure of
$$C_1 + C_2 = \{x + y : x \in C_1, \; y \in C_2\}$$
is also compact.

6. Prove that if H is a Hilbert space, K is a compact symmetric operator on H, and Y is a closed subspace of X, then the map $K|_Y$ is compact.

7. Suppose K is a bounded compact symmetric operator with non-negative eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots$ and corresponding eigenvectors $z_1, \ldots, z_n$. Prove that for each n,
$$\lambda_n = \max_{x \perp z_1, \ldots, z_{n-1}} \frac{\langle Kx.x \rangle}{\|x\|^2}.$$
(This is known as the *Rayleigh principle*.)

8. Let K be a compact bounded symmetric operator and let $z_1, \ldots, z_n$ be eigenvectors with corresponding eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let X be the linear subspace spanned by $\{z_1, \ldots, z_n\}$. Prove that if $y \in X$, then $\langle Ky, y \rangle \geq \lambda_n \langle y, y \rangle$.

9. Prove that the nth largest non-negative eigenvalue for a compact bounded symmetric operator K satises
$$\lambda_n = \max \left\{ \min_{x \in S_n} \frac{\langle Kx, x \rangle}{\|x\|^2} : \ S_n \text{ is a linear subspace of dimension } n. \right\}$$
(This is known as *Fisher's principle*.)

10. Prove that the nth largest non-negative eigenvalue for a compact bounded symmetric operator K satises
$$\lambda_n = \min \left\{ \max_{x \in S_{n-1}^{\perp}} \frac{\langle Kx, x \rangle}{\|x\|^2} : \ S_{n-1} \text{ is a linear subspace of dimension } n-1. \right\}$$
(This is known as *Courant's principle*.)

11. Say that A is a *positive operator* if $\langle Ax, x \rangle \geq 0$ for all x. (In the case of matrices, the term used is *positive semidefinite*.) Suppose A and B are compact symmetric operators and that $A - B$ is also a positive operator. Suppose A and B have eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots$ and $\beta_1 \geq \beta_2 \geq \cdots$, respectively, arranged in decreasing order. Prove that $\alpha_k \geq \beta_k$ for all k.

12. Let A be a compact symmetric operator. Find necessary and sufficient conditions on a continuous function f such that $f(A)$ is a compact symmetric operator.

13. Let A be a bounded symmetric operator. For $z, w \in \sigma(A)^c$, put $R_z \stackrel{\text{def}}{=} (z-A)^{-1}$ and $R_w \stackrel{\text{def}}{=} (w-A)^{-1}$. Prove the *resolvent identity*:
$$R_w - R_z = (z - w) R_w R_z.$$

14. Suppose A is a bounded symmetric operator and f is a continuous function on $\sigma(A)$. Let P_n be polynomials which converge uniformly to f on $\sigma(A)$. Suppose $\lambda \in \mathbf{C}$ and suppose that there exists $\epsilon > 0$ such that $d(\lambda, \sigma(P_n(A))) \geq \epsilon$ for each n. Prove that $\lambda \notin \sigma(f(A))$.

15. Let H be a separable Hilbert space and suppose that $K : H \to H$ is a compact symmetric linear operator. Prove that K is a positive operator if and only if all the eigenvalues of K are non-negative.

16. Let A be a bounded symmetric operator, not necessarily compact. Prove that if $A = B^2$ for some bounded symmetric operator B, then A is a positive operator.

17. Let A be a bounded symmetric operator whose spectrum is contained in $[0, \infty)$. Prove that A has a positive square root, namely that there exists a bounded positive symmetric operator B such that $A = B^2$.

18. Let A be a bounded symmetric operator, not necessarily compact. Prove that A is a positive operator if and only if $\sigma(A) \subset [0, \infty)$.

19. Let A be a bounded symmetric operator. Prove that $\mu_{x,x}$ is a real non-negative measure.

20. Suppose that A is a bounded symmetric operator, $C_1, \dots, C_n$ are disjoint Borel measurable subsets of $\mathbf{C}$, and $a_1, \dots, a_n$ are complex numbers. Prove that

$$\left\| \sum_{k=1}^{n} a_k E(C_k) \right\| = \max_{1 \le k \le n} |a_k|.$$

21. Prove that if A is a bounded symmetric operator and f is a bounded Borel measurable function, then

$$\|f(A)x\|^2 = \int_{\sigma(A)} |f(z)|^2 \, d\mu_{x,x}(z).$$

22. Prove that if A is a bounded symmetric operator and $\{s_n\}$ is a sequence of simple functions such that s_n converges uniformly to a Borel measurable function f on $\sigma(A)$, then $\|f(A) - s_n(A)\| \to 0$.