Math 5052
Measure Theory and Functional Analysis II
Homework Assignment 11

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Due Friday, April 15, 2016

Read Chapters 23 (Sobolev spaces) and 26 (Distributions) in the textbook.

Please do Exercises 4, 5, 6, 7, 13, 15, 18, 22, 23, 25.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. Some of these exercises originate from that source.

1. Prove that if $p_1, \ldots, p_n > 1$ with
$$\sum_{i=1}^{n} \frac{1}{p_i} = 1,$$
and μ is a σ-finite measure, then
$$\int |f_1 \cdots f_n| \, d\mu \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$
This is known as the *generalized Hölder inequality.*

2. Suppose $1 \leq p < \infty$ and $f \in L^p$. Prove that if there exists a sequence $\{f_m\}$ such that

   1. $(\forall m) f_m \in C_K^\infty$,

   2. $\|f - f_m\|_p \to 0$ as $m \to \infty$,

   3. for all $|j| \leq k$, $D^j f_m$ converges in L^p,

   then $f \in W^{k,p}$.

3. Suppose $1 \leq p < \infty$ and $f \in W^{k,p}$. Prove that there exists a sequence $\{f_m\}$ such that

   1. $(\forall m) f_m \in C_K^\infty$,

   2. $\|f - f_m\|_p \to 0$ as $m \to \infty$,

3. for all $|j| \le k$, $D^j f_m$ converges in L^p.

4. Suppose $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Prove that

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

This is known as *Young's inequality.*

5. Prove that $W^{k,2}$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle \overset{\text{def}}{=} \int (1 + |u|^2)^k \hat{f}(u)\, \overline{\hat{g}(u)}\, du.$$

Here $\hat{f}, \hat{g}$ denote the Fourier transforms of f, g respectively.

Hint: use Plancherel's theorem.

6. For real number s, define

$$H^s \overset{\text{def}}{=} \{f : \int (1 + |u|^2)^s |\hat{f}(u)|^2\, du < \infty\}.$$

Prove that if $s > n/2$, then $\hat{f}$ belongs to L^1 and thus f is continuous.

Note: this is a particular case of the *Sobolev embedding theorem.*

7. Does the product rule hold for weak derivatives? That is, if $p \ge 2$ and $f, g \in W^{1,p}$, is $fg \in W^{1,p/2}$ with $D(fg) = (Df)g + f(Dg)$? Prove or give a counterexample.

Solution: One might be tempted to say "no" and seek a counterexample of the form $f = g = \chi_{(0,1)}$, which has $D(fg) = Df = Dg = \delta_0 - \delta_1$ but apparently $fDg + gDf = 0$. However, this example does not work because the Dirac masses δ_0 and δ_1 do not belong to L^p for any $p > 0$. Consequently, the products fDg and gDf are not well-defined and cannot be evaluated by pointwise multiplication.

The answer is in fact "yes," the product rule does hold for weak derivatives in the Sobolev space $W^{1,p}$ with $p \ge 2$. A key difference with distributions is that functions in the Sobolev space $W^{1,p}$ can be approximated in norm by test functions. Consequently, their products are well defined.

So, given $f, g \in W^{1,p}$, let $\phi_k \to g$ be a norm convergent sequence of test functions, namely, $\phi_k \in C_K^\infty$ for all k and $\|\phi_k - g\|_{W^{1,p}} \to 0$ as $k \to \infty$. In particular, this means $\|\phi_k - g\|_p \to 0$ as $k \to \infty$ and $\|\phi_k' - Dg\|_p \to 0$ as $k \to \infty$, where Dg is the weak derivative of g and ϕ_k' is the usual continuous derivative of ϕ_k.

Since C_K^∞ is preserved under multiplication, the product rule holds for weak derivatives of the products $f\phi_k$:

$$D(f\phi_k) = \phi_k Df + f D\phi_k = \phi_k Df + f\phi_k',$$

in the sense of L^p, since the weak derivative $D\phi_k$ and the usual derivative ϕ_k' agree a.e.

One may now use Hölder's inequality: $\|fg\|_c \le \|f\|_a \|g\|_b$ whenever $\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$. (This generalization follows immediately by taking $|f|^c$ and $|g|^c$ instead of f and g in the basic Hölder inequality for $1 = \frac{1}{a/c} + \frac{1}{b/c}$.) Put $a = b = p/2$ and $c = p$ to get

$$\|(\phi_k - g)Df\|_{p/2} \le \|\phi_k - g\|_p \|Df\|_p \to 0 \quad \text{and} \quad \|(\phi_k' - Dg)f\|_{p/2} \le \|\phi_k' - Dg\|_p \|f\|_p \to 0,$$

as $k \to \infty$.

Now $p \ge 2$ insures that $\|\cdot\|_{p/2}$ satisfies the triangle inequality, so by the linearity of the weak derivative,

$$\|D(f\phi_k) - fDg - gDf\|_{p/2} \le \|(\phi_k - g)Df\|_{p/2} + \|(\phi_k' - Dg)f\|_{p/2} \to 0,$$

as $k \to \infty$.

Since $\|f\phi_k - fg\|_{p/2} \to 0$ by a similar Hölder inequality argument, conclude that $fg \in L^{p/2}$ and that $D(fg) = fDg + gDf$ as claimed. $\square$

8. Suppose that $k \ge 1$, $p < n/k$, and q satisfies $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. Prove that there exists c such that

$$\|f\|_q \le c \left\| \sum_{|j| \le k} |D^k f| \right\|_p .$$

(This is theorem 23.5 on textbook p.322.)

9. Suppose that $\psi : \mathbf{R}^2 \to \mathbf{R}$ is a C_K^1 function that equals 1 on $B(0,1) \subset \mathbf{R}^2$. Define

$$f(x_1, x_2) \overset{\text{def}}{=} \psi(x_1, x_2) \frac{x_1^2}{x_1^2 + x_2^2}.$$

Prove that $f \in W^{1,p}(\mathbf{R}^2)$ for $1 \le p < 2$, but that f is not continuous.

Note: the function ψ is introduced only to insure that f has compact support.

10. Prove that if $f \in W^{1,1}(\mathbf{R})$, then f is continuous.

11. Prove that if $f \in W^{1,p}(\mathbf{R})$ for some $p > 1$, then f is Hölder continuous, namely, there exists $c > 0$ and $\alpha \in (0,1)$ such that $|f(x) - f(y)| \le c|x - y|^\alpha$ for all $x, y \in \mathbf{R}$.

12. Prove that if $f \in C_K^1$, then

$$f(x) = c_1^{-1} \sum_{j=1}^n \int \frac{\partial f}{\partial x_j}(x - y) \frac{y_j}{|y|^n}\, dy,$$

where c_1 is equal to the surface measure of $\partial B(0,1)$.

13. Suppose $n \ge 3$. Prove the *Nash inequality*:

$$\left(\int |f|^2 \right)^{1+2/n} \le c_1 \left(\int |\nabla f|^2 \right) \left(\int |f| \right)^{4/n}$$

if $f \in C^1_K(\mathbf{R}^n)$, where the constant c_1 depends only on n.

Note: the Nash inequality is also true when $n = 2$.

14. Can C^∞_K be given a metric topology such that convergence in C^∞_K is convergence in the metric? If so, is the resulting metric space complete? If so, can the metric be chosen so that C^∞_K is a Banach space?

15. Define a function on $\mathcal{S} \times \mathcal{S}$ by setting

$$d(f,g) \stackrel{\text{def}}{=} \sum_{k,j} \frac{1}{2^{j+k}} \frac{\|f-g\|_{j,k}}{1+\|f-g\|_{j,k}}.$$

  a. Prove that this is a metric for the Schwartz class $\mathcal{S}$.

  b. Prove that a sequence in $\mathcal{S}$ converges in the Schwartz class sense if and only if it converges in the metric d.

  c. Is $\mathcal{S}$ with this metric a complete metric space?

16. Suppose that $f \in C^\infty_K$ and let

$$F(f) = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{f(x)}{x}\, dx.$$

  a. Prove that the limit exists.

  b. Prove that F is a distribution.

17. Suppose that $U : C^\infty_K \to C^\infty_K$ is a continuous linear map. For a distribution F, define

$$TF(f) \stackrel{\text{def}}{=} F(Uf), \qquad f \in C^\infty_K.$$

  a. Prove that TF is a distribution.

  b. Suppose that $V : C^\infty_K \to C^\infty_K$ is a continuous linear map such that $\int g(Uf) = \int (Vg)f$ for all $f, g \in C^\infty_K$. Prove that if $\in C^\infty_K$, then

$$TG_g = G_{Vg}.$$

18. If μ is a finite measure defined on the Borel σ-algebra, prove that F given by $F(f) \stackrel{\text{def}}{=} \int f\, d\mu$ is a distribution.

19. Suppose g is a continuously differentiable function and $g' = h$ is its (classical) derivative. Prove that $DG_g = G_h$.

20. A *positive distribution* F is one such that $F(f) \geq 0$ whenever $f \geq 0$. Prove that if K is a compact set and F is a positive distribution, then there exists a constant c such that

$$|F(f)| \leq c \sup_{x \in K} |f(x)|,$$

for all f with $\operatorname{supp} f \subset K$.

21. Prove that if F is a positive distribution (see exercise 20 above) and has compact support, then there exists a measure μ such that $F(f) = \int f \, d\mu$ for all $f \in C_K^\infty$.

22. Suppose
$$F(f) = \lim_{\epsilon \to 0} \int_{1 \geq |x| \geq \epsilon} \frac{f(x)}{x} \, dx.$$

   a. Show that F is a distribution with compact support.

   b. Prove that F has the representation $F = \sum_{j \leq L} D^j G_{g_j}$ of eq.26.3 on textbook p.388.

   c. Find explicit values of L and g_j for this F, as in theorem 26.15 on textbook page 388.

23. Let $g_1(x) = e^x$ and $g_2(x) = e^x \cos(e^x)$. Prove that G_{g_2} is a tempered distribution but G_{g_1} is not.

24. Prove eq.26.5 on textbook p.319, namely
$$u^j D^k (\mathcal{F} f)(u) = i^{k+j} \mathcal{F}(D^j(x^k f))(u),$$
where $\mathcal{F}$ is the Fourier transform.

25. Determine $\mathcal{F} G_1$, $\mathcal{F}\delta$, and $\mathcal{F} D^j \delta$ for $j \geq 1$.