Math 5052
Measure Theory and Functional Analysis II
Homework Assignment 12

Prof. Wickerhauser

Due **Monday, May 2**, 2016

Read Chapter 21 (Probability) in the textbook.

Please do Exercises 4, 9, 15*, 18, 22, 26, 35*, 38, 39, 40*

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. Some of these exercises originate from that source.

1. Show that if X has a continuous distribution function F_X and $Y = F_X(X)$, then Y has a density $f_Y(x) = \mathbf{1}_{[0,1]}(x)$.

2. Find an example of a probability space and three events A, B, and C such that $\Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C)$, but A, B, and C are not independent events.

3. Suppose that
$$\Pr(X \leq x, Y \leq y) = \Pr(X \leq x)\Pr(Y \leq y)$$
for all $x, y \in \mathbf{R}$. Prove that X and Y are independent random variables.

4. Find a sequence of events $\{A_n\}$ such that
$$\sum_{n=1}^{\infty} \Pr(A_n) = \infty$$
but $\Pr(A_n \text{ i.o.}) = 0$.

5. A random vector $X = (X_1, \ldots, X_n)$ has a *joint density* f_X if $\Pr(X \in A) = \int_A f_X(x)\,dx$ for all Borel subsets A of $\mathbf{R}^n$. Here the integral is with respect to n dimensional Lebesgue measure.

   a. Prove that if the joint density of X factors into the product of densities of the X_j, namely $f_X(x) = \prod_{j=1}^{n} f_j(x_j)$, for almost every $x = (x_1, \ldots, x_n)$, then the X_j are independent.

b. Prove that if X has a joint density and the X_j are independent, then each X_j has a density and the joint density of X factors into the product of the densities of the X_j.

6. Suppose $\{A_n\}$ is a sequence of events, not necessarily independent, such that $\sum_{n=1}^{\infty} \Pr(A_n) = \infty$. Suppose in addition that there exists a constant c such that for each $N \geq 1$,

$$\sum_{i,j=1}^{N} \Pr(A_n \text{ i.o.}) \leq c \left(\sum_{i=1}^{N} \Pr(A_i) \right)^2 .$$

Prove that $\Pr(A_n \text{ i.o.}) > 0$.

7. Suppose X and Y are independent, $\mathrm{E}|X|^p < \infty$ for some $p \in [1, \infty)$, $\mathrm{E}|Y| < \infty$, and $\mathrm{E}Y = 0$. Prove that

$$\mathrm{E}(|X+Y|^p) \geq \mathrm{E}|X|^p.$$

8. Suppose that X_i are independent random variables such that $\mathrm{Var}X_i/i \to 0$ as $i \to \infty$. Suppose also that $\mathrm{E}X_i \to a$. Prove that S_n/n converges in probability to a, where $S_n = \sum_{i=1}^{n} X_i$. (It is not assumed that the X_i are identically distributed.)

9. Suppose $\{X_i\}$ is a sequence of independent mean zero random variables, not necessarily identically distributed. Suppose that $\sup_i \mathrm{E}X_i^4 < \infty$.

   a. If $S_n \overset{\text{def}}{=} \sum_{i=1}^{n} X_i$, prove that there exists a constant c such that $\mathrm{E}S_n^4 \leq cn^2$.

   b. Prove that $S_n/n \to 0$ a.s.

10. Suppose $\{X_i\}$ is an i.i.d. sequence such that S_n/n converges a.s., where $S_n \overset{\text{def}}{=} \sum_{i=1}^{n} X_i$.

   a. Prove that $X_n/n \to 0$ a.s.

   b. Prove that $\sum_n \Pr(|X_n| > n) < \infty$.

   c. Prove that $\mathrm{E}|X_1| < \infty$.

11. Suppose $\{X_i\}$ is an i.i.d. sequence of random variables with $\mathrm{E}|X_1| < \infty$.

   a. Prove that the sequence $\{S_n/n\}$ is uniformly integrable by the definition in Exercise 7.16 on textbook p.58.

   b. Prove that $\mathrm{E}S_n/n$ converges to $\mathrm{E}X_1$.

12. Suppose $\{X_i\}$ is an i.i.d. sequence of random variables with $\mathrm{E}|X_1| < \infty$ and $\mathrm{E}X_1 = 0$. Prove that

$$\frac{\max_{1 \leq k \leq n} |S_k|}{n} \to 0, \qquad \text{a.s.}$$

13. Suppose that $\{X_i\}$ is a sequence of independent random variables with mean zero such that $\sum_i \mathrm{Var} X_i < \infty$. Prove that S_n converges a.s. as $n \to \infty$, where $S_n \stackrel{\text{def}}{=} \sum_{i=1}^n X_i$.

14. Let $\{X_i\}$ be a sequence of random variables. The *tail σ-field* is defined to be

$$\mathcal{T} \stackrel{\text{def}}{=} \bigcap_{n \geq 1} \sigma(X_n, X_{n+1}, \ldots).$$

Let $S_n \stackrel{\text{def}}{=} \sum_{i=1}^n X_i$.

  a. Prove that the event (S_n converges) is in $\mathcal{T}$.

  b. Prove that the event $(S_n/n > a)$ is in $\mathcal{T}$ for each real number a.

15. Let $\{X_i\}$ be a sequence of independent random variables and let $\mathcal{T}$ be the tail σ-field.

  a. Prove that if $A \in \mathcal{T}$, then A is independent of $\sigma(X_1, \ldots, X_n)$ for each n.

  b. Prove that if $A \in \mathcal{T}$, then A is independent of itself, and hence $\Pr(A)$ is either 0 or 1.

Note: part b is known as the *Kolmogorov 0-1 law.*

16. Let $\{X_i\}$ be an i.i.d. sequence of random variables. Prove that if $\mathrm{E}X_1^+ = \infty$ and $\mathrm{E}X_1^- < \infty$, then $S_n/n \to +\infty$ a.s., where $S_n \stackrel{\text{def}}{=} \sum_{i=1}^n X_i$.

17. Let $\mathcal{F} \subset \mathcal{G}$ be two σ-fields. Let H be the Hilbert space of $\mathcal{G}$-measurable random variables Y such that $\mathrm{E}Y^2 < \infty$ and let M be the subspace of H consisting of the $\mathcal{F}$-measurable random variables. Prove that if $Y \in H$, then $\mathrm{E}[Y|\mathcal{F}]$ is equal to the orthogonal projection of Y onto the subspace M.

18. Suppose $\mathcal{F} \subset \mathcal{G}$ are two σ-fields and X and Y are bounded $\mathcal{G}$-measurable random variables. Prove that

$$\mathrm{E}[X\mathrm{E}[Y|\mathcal{F}]] = \mathrm{E}[Y\mathrm{E}[X|\mathcal{F}]].$$

19. Let $\mathcal{F} \subset \mathcal{G}$ be two σ-fields and let X be a bounded $\mathcal{G}$-measurable random variables. Prove that if

$$\mathrm{E}[XY] = \mathrm{E}[X\mathrm{E}[Y|\mathcal{F}]]$$

for all bounded $\mathcal{G}$-measurable random variables Y, then X is $\mathcal{F}$-measurable.

20. Suppose $\mathcal{F} \subset \mathcal{G}$ are two σ-fields and that X is $\mathcal{G}$-measurable with $\mathrm{E}X^2 < \infty$. Set $Y \stackrel{\text{def}}{=} \mathrm{E}[X|\mathcal{F}]$. Prove that if $\mathrm{E}X^2 = \mathrm{E}Y^2$, then $X = Y$ a.s.

21. Suppose $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_N$ are σ-fields. Suppose A_i is a sequence of random variables adapted to $\{\mathcal{F}_i\}$ such that $A_1 \leq A_2 \leq \cdots$ and $A_{i+1} - A_i \leq 1$ a.s. for each i. Prove that if $\mathrm{E}[A_N - A_i|\mathcal{F}] \leq 1$ a.s. for each i, then $\mathrm{E}A_N^2 < \infty$.

22. Let $\{X_i\}$ be an i.i.d. sequence of random variables with $\Pr(X_1 = 1) = \Pr(X_1 = -1) = \frac{1}{2}$. Let $S_n \stackrel{\text{def}}{=} \sum_{i=1}^n X_i$. This sequence $\{S_n\}$ is called a *simple random walk*. Let

$$L \stackrel{\text{def}}{=} \max\{k \leq 9 : S_k = 1\} \wedge 9.$$

Prove that L is *not* a stopping time with respect to the family of σ-fields $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$.

23. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ be an increasing family of σ-fields and let $\mathcal{F}_\infty \stackrel{\text{def}}{=} \sigma(\cup_n \mathcal{F}_n)$. If N is a stopping time, define

$$\mathcal{F}_N \stackrel{\text{def}}{=} \{A \in \mathcal{F}_\infty : A \cap (N \leq n) \in \mathcal{F}_n \text{ for all } n.\}.$$

a. Prove that $\mathcal{F}_N$ is a σ-field.

b. If M is another stopping time with $M \leq N$ a.s., and $\mathcal{F}_M$ is defined analogously, prove that $\mathcal{F}_M \subset \mathcal{F}_N$.

c. If X_n is a martingale with respect to $\{\mathcal{F}_n\}$ and N is a stopping time bounded by the real number K, prove that $\mathrm{E}[X_n|\mathcal{F}_N] = X_N$.

24. Let $\{X_i\}$ be a sequence of bounded i.i.d. random variables with mean 0. Let $S_n = \sum_{i=1}^n X_i$.

a. Prove that there exists a constant c_1 such that $M_n \stackrel{\text{def}}{=} e^{S_n - c_1 n}$ is a martingale.

b. Show that there exists a constant c_2 such that

$$\Pr(\max_{1 \leq k \leq n} S_k > \lambda) \leq 2e^{-c_2 \lambda^2 / n}$$

for all $\lambda > 0$.

25. Let $\{X_i\}$ be a sequence of i.i.d. standard normal random variables with mean 0. Let $S_n = \sum_{i=1}^n X_i$.

a. Prove that for each $a > 0$, $M_n \stackrel{\text{def}}{=} e^{aS_n - a^2 n/2}$ is a martingale.

b. Show that

$$\Pr(\max_{1 \leq k \leq n} S_k > \lambda) \leq e^{-\lambda^2 / 2n}$$

for all $\lambda > 0$.

26. Let $\{X_n\}$ be a submartingale. Let

$$A_n \stackrel{\text{def}}{=} \sum_{i=2}^{n} (X_i - \mathrm{E}[X_i|\mathcal{F}_{i-1}]).$$

Prove that $M_n \stackrel{\text{def}}{=} X_n - A_n$ is a martingale.

Note: this is known as the *Doob decomposition* of a submartingale.

27. Suppose M_n is a martingale. Prove that if

$$\sup_n \mathrm{E}M_n^2 < \infty,$$

then M_n converges a.s. and also in L^2.

28. Set $(\Omega, \mathcal{F}, \mathrm{P})$ equal to $([0,1], \mathcal{B}, m)$, where $\mathcal{B}$ is the Borel σ-field on $[0,1] \subset \mathbf{R}$ and m is Lebesgue measure. Define

$$X_n(\omega) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \omega \in \left[\frac{2k}{2n}, \frac{2k+1}{2^n}\right) \text{ for some } k \leq 2^{n-1}; \\ -1, & \text{if } \omega \in \left[\frac{2k+1}{2n}, \frac{2k+2}{2^n}\right) \text{ for some } k \leq 2^{n-1}. \end{cases}$$

a. Prove that X_n converges weakly (in the probabilistic sense) to a nonzero random variable.

b. Prove that X_n converges to 0 in the weak $L^2(m)$ sense, namely $\mathrm{E}[X_n Y] \to 0$ for all $Y \in L^2$.

29. Suppose X_n is a sequence of random variables that converges weakly to a random variable X. Prove that the sequence $\{X_n\}$ is tight.

Note: see textbook p.275 for the definition of *tight*.

30. Suppose $X_n \to X$ weakly and $Y_n \to 0$ in probability. Prove that $X_n Y_n \to 0$ in probability.

31. Given two probability measures P and Q on $[0,1]$ with the Borel σ-field, define

$$d(\mathrm{P}, \mathrm{Q}) \stackrel{\text{def}}{=} \sup\{\left|\int f\, d\mathrm{P} - \int f\, d\mathrm{Q}\right| : f \in C^1, \|f\|_\infty \leq 1, \|f'\|_\infty \leq 1\}.$$

Here C^1 is the collection of continuously differentiable functions $f : \mathbf{R} \to \mathbf{R}$, and f' is the derivative of f.

a. Prove that d is a metric.

b. Prove that $\mathrm{P}_n \to \mathrm{P}$ weakly if and only if $d(\mathrm{P}_n, \mathrm{P}) \to 0$.

Note: This metric makes sense only for probabilities defined on $[0,1]$. There are other metrics for weak convergence that work in more general situations.

32. Suppose $F_n \to F$ weakly and every pointof F is a continuity point. Prove that F_n converges to F uniformly over $x \in \mathbf{R}$:

$$\sup_{x \in \mathbf{R}} |F_n(x) - F(x)| \to 0.$$

33. Suppose $X_n \to X$ weakly. Prove that ϕ_{X_n} converges uniformly to ϕ_X on each bounded interval.

34. Suppose X_n is a collection of random variables that is tight. Prove that $\{\phi_{X_n}\}$ is equicontinuous on $\mathbf{R}$.

35. Suppose $X_n \to X$ weakly, $Y_n \to Y$ weakly, and X_n and Y_n are independent for each n. Prove that $X_n + Y_n \to X + Y$ weakly.

36. X is said to be a *gamma* random variable with parameters λ and t if X has density

$$\frac{1}{\Gamma(x)}\lambda^t x^{t-1} e^{-\lambda t}\mathbf{1}_{(0,\infty)}(x),$$

where $\Gamma(t) \stackrel{\text{def}}{=} \int_0^\infty y^{t-1}e^{-t}\,dy$ is the Gamma function.

a. Prove that an exponential random variable with parameter λ is also a gamma random variable with parameters 1 and λ.

b. Prove that if X is a standard normal random variable, then X^2 is a gamma random variable with parameters $1/2$ and $1/2$.

c. Find the characteristic function of a gamma random variable.

d. Prove that if X is a gamma random variable with parameters t and λ, and X and Y are independent, then $X + Y$ is also a gamma random variable. Determine the parameters of $X + Y$.

37. Suppose X_n is a sequence of independent random variables, not necessarily identically distributed, with $\sup_n \mathrm{E}|X_n|^3 < \infty$ and $\mathrm{E}X_n = 0$ and $\mathrm{Var}X_n = 1$ for each n. Prove that $S_n/\sqrt{n}$ converges weakly to a standard normal random variable, where $S_n \stackrel{\text{def}}{=} \sum_{i=1}^n X_n$.

38. Suppose that X_n is a Poisson random variable with parameter n for each n. Prove that $(X_n - n)/\sqrt{n}$ converges weakly to a standard normal random variable as $n \to \infty$.

39. Suppose that P is a probability measure on the Borel subsets of $\mathbf{R}^n$.

a. For $\omega = (\omega_1, \dots, \omega_n) \in \mathbf{R}^n$, define $X_k(\omega) \stackrel{\text{def}}{=} \omega_k$ for $k = 1, \dots, n$ and put $X = (X_1, \dots, X_n)$. Prove that the law P_X of X is equal to P.

b. If P is a product measure, prove that the components of X are independent.

40. Prove that if X_t is a Brownian motion and a is a nonzero real number, then $Y_t \stackrel{\text{def}}{=} aX_{a^2t}$ is also a Brownian motion.

41. Let X_t be a Brownian motion. Fix $n \geq 1$ and let $M_k \stackrel{\text{def}}{=} X_{k/2^n}$.

a. Prove that M_k is a martingale.

b. Prove that if $a \in \mathbf{R}$, then $e^{aM_k - a^2(k/2^n)/2}$ is a martingale.

c. Prove that

$$\mathrm{P}\left(\sup_{t\leq r} X_t \geq \lambda\right) \leq e^{-\lambda^2/2r}.$$

42. Let X_t be a Brownian motion. Let

$$A_n \stackrel{\text{def}}{=} \left(\sup_{t \le 2^{n+1}} X_t > \sqrt{4 \cdot 2^n \log\log 2^n} \right).$$

a. Prove that $\sum_{n=1}^{\infty} \Pr(A_n) < \infty$.

b. Prove that

$$\limsup_{t\to\infty} \frac{X}{\sqrt{t \log\log t}} < \infty, \qquad \text{a.s.}$$

Note: these results are part of what is called the *law of the interated logarithm* for Brownian motion.

43. Let X_t be a Brownian motion. Let $M > 0$, $t_0 > 0$, and

$$B_n \stackrel{\text{def}}{=} \left(X_{t_0+2^{-n}} - X_{t_0+2^{-n-1}} > M2^{-n-1} \right).$$

a. Prove that $\sum_{n=1}^{\infty} \Pr(B_n) = \infty$.

b. Prove that the function $t \mapsto X_t(\omega)$ is not differentiable at $t = t_0$.

c. Prove that except for ω in a null set, the function $t \mapsto X_t(\omega)$ is not differentiable at almost every t with respect to Lebesgue measure on $[0, \infty)$.

Note: it can be shown by a different proof that the function $t \mapsto X_t(\omega)$ is nowhere differentiable except for ω in a null set.

44. Let X_t be a Brownian motion and let $h > 0$ be given. Prove that except for ω in a null set, there are times $t \in (0, h)$ for which $X_t(\omega) > 0$. (These times will depend on ω.) Conclude that almost every Brownian path must oscillate quite a bit near 0.

45. Let X_t be a Brownian motion on $[0, 1]$ Prove that $Y_t \stackrel{\text{def}}{=} X_1 - X_{1-t}$ is also a Brownian motion.