Partitions of Unity.

Recall that a topological space \((X, T)\) is \textit{compact} if every open cover of \(X\) contains a finite subcover. This is a strong condition, but there are several related weaker conditions of interest.

1.1 Closures

Set \(F \subset X\) is called \textit{closed} iff its complement \(F^c \overset{\text{def}}{=} X \setminus F\) is open.

The following are de Morgan’s formulas. Index set \(I\) is arbitrary:

\[
(\bigcap_{\alpha \in I} S_{\alpha})^c = \bigcup_{\alpha \in I} S_{\alpha}^c, \quad (\bigcup_{\alpha \in I} S_{\alpha})^c = \bigcap_{\alpha \in I} S_{\alpha}^c.
\]

Use these to deduce that

- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

Define the \textit{closure} \(\bar{S}\) of a subset \(S \subset X\) to be the smallest closed set that contains \(S\), namely

\[
\bar{S} = \bigcap_{\{F: F^c \in T, S \subset F\}} F
\]

which is the intersection of all the closed sets \(F\) (whose complements are open sets \(F^c \in T\)) that contain \(S\).

Any closed subset \(F\) of a compact set \(K\) is compact: if \(F \subset \bigcup_{\alpha \in I} G_{\alpha}\) is any open cover, then \(K \subset F^c \cup [\bigcup_{\alpha \in I} G_{\alpha}]\) is an open cover of \(K\), hence it has a finite subcover \(F^c \cup G_1 \cup \cdots \cup G_N\), hence \(F \subset G_1 \cup \cdots \cup G_N\) is a finite subcover of \(F\).

Consequently, if \(K\) is compact and \(S \subset K\) also has \(\bar{S} \subset K\), then \(\bar{S}\) is compact.

A compact subset \(K\) need not be closed. For example, let \(X = \{a, b\}\) and \(T = \{\emptyset, \{a\}, \{a, b\}\}\). Then \(\{a\}\) is compact but not closed, since its complement \(\{a\}^c = \{b\} \notin T\). (There are many other similarly contrived examples.)
However, any compact subset \( K \) of a Hausdorff space must be closed. Prove this by showing that its complement \( K^c \) is open, for which it suffices to show that each \( x \in K^c \) belongs to some open set \( G \subset K^c \). Find that \( G \) as follows:

- For each \( y \in K \), find disjoint open \( G_y, H_y \) with \( x \in G_y \) and \( y \in H_y \). These exist by the Hausdorff property.
- Note that \( K \subset \bigcup_{y \in K} H_y \) is an open cover.
- Since \( K \) is compact, there exists a finite subcover which may be denoted \( K \subset H_1 \cup \cdots \cup H_N \).
- Let \( G_i \) be the open set around \( x \) corresponding to \( H_i, i = 1, \ldots, N \).
- Put \( G \defeq G_1 \cap \cdots \cap G_N \). This finite intersection of open sets is open.
- Note that \( x \in G \).
- \( G \subset K^c \), since \((\forall i) G \cap H_i = \emptyset \) and \( K \subset \cup H_i \).

Conclude that \( K^c \) is open, so \( K \) is closed.

### 1.2 Local Compactness

A topological space \((X, \mathcal{T})\) is **locally compact** if every point \( x \in X \) has a compact neighborhood.

A \( d \)-dimensional manifold \( \mathcal{M} \) is locally compact since it is a metric space that is locally homeomorphic to \( \mathbb{E}^d \). Every \( x \in \mathcal{M} \) belongs to some chart \((G, \phi)\) with open \( G \), so there exists \( \epsilon > 0 \) such that \( x \in B(x, 2\epsilon) \subset G \). But then

\[
\bar{B}(x, \epsilon) \defeq \{ y \in \mathcal{M} : d(x, y) \leq \epsilon \} \subset G,
\]

and \( \phi(\bar{B}(x, \epsilon)) \subset \phi(G) \subset \mathbb{E}^d \) is closed and bounded. Such sets are compact by the Heine-Borel theorem. Conclude that

\[
\bar{B}(x, \epsilon) = \phi^{-1}\left( \phi(\bar{B}(x, \epsilon)) \right) \subset \mathcal{M}
\]

is a compact neighborhood of \( x \).

### 1.3 Paracompactness

Start with the notion of **refinement** of open covers: collection \( \mathcal{G}' \subset \mathcal{T} \) is a refinement of \( \mathcal{G} \subset \mathcal{T} \) iff

\[
(\forall G \in \mathcal{G})(\exists G' \in \mathcal{G}')G' \subset G.
\]

For example, in a metric space \( X \), \( \mathcal{G}' = \{B(x, \epsilon/2) : x \in X\} \) is a refinement of \( \mathcal{G} = \{B(x, \epsilon) : x \in X\} \).
Next, say that an open cover \( \{ G_\alpha : \alpha \in I \} \) is \textit{locally finite} iff every \( x \in X \)
is contained in some neighborhood \( U_x \) that intersects only finitely many sets in the cover. Namely,
\[
(\forall x \in X)(\exists U_x \in \mathcal{T}) (x \in U_x \text{ and } \{ \alpha \in I : U_x \cap G_\alpha \neq \emptyset \} \text{ is finite}).
\]

Finally, say that \((X, \mathcal{T})\) is \textit{paracompact} iff every open cover has a locally finite refinement.

\textbf{Remark.} If there is a locally finite refinement \( \mathcal{G}' = \{ G'_\beta : \beta \in J \} \) of \( \mathcal{G} = \{ G_\alpha : \alpha \in I \} \), then there is an identically-indexed locally finite refinement \( \mathcal{H} = \{ H_\alpha : \alpha \in I \} \) of \( \mathcal{G} \) such that \( H_\alpha \subset G_\alpha \) for each \( \alpha \in I \). It may be constructed by choosing a single \( \alpha = i(\beta) \in I \) for each \( \beta \in J \) such that \( G'_\beta \subset G_\alpha \), and then putting
\[
H_\alpha = \bigcup_{\{ \beta \in J : i(\beta) = \alpha \}} G'_\beta.
\]
It is clear that \( H_\alpha \subset G_\alpha \). But also, any neighborhood that intersects only finitely many \( G'_\beta \) can intersect at most finitely many \( H_\alpha \). (Since \( i \) is single-valued, there will be more intersecting \( \beta \) indices than \( \alpha \) indices.) Thus \( \mathcal{H} \) is a locally finite refinement of \( \mathcal{G} \).

\textbf{Theorem.} A locally compact second countable Hausdorff space is paracompact.

\textbf{Proof.} See Proposition A1.6 in \textit{PartOfUnity_LocFiniteRefinements.pdf} on the class website. \( \square \)

\textbf{Remark.} A differentiable manifold is locally compact (because it is finite dimensional), second countable (by definition), and Hausdorff (because it is a metric space). Hence any differentiable manifold is paracompact.

\textbf{Remark.} Any compact differentiable manifold is obviously paracompact since any finite cover is locally finite.

\section*{1.4 Bump functions}

Start by constructing a continuously differentiable nonnegative function \( b : \mathbb{E}^d \to \mathbb{R} \) satisfying
\begin{itemize}
  \item \(|x| \geq 2 \implies b(x) = 0\). Namely, \( b \) is supported in \( B(0,2) \), the closed ball of radius 2 centered at \( 0 \).
  \item \(|x| \leq 1 \implies b(x) > 0\). Thus \( b \) is strictly positive in \( B(0,1) \).
\end{itemize}
Examples among elementary functions include
\[
b(x) = \begin{cases} 
  \frac{1}{2}[1 + \cos(\pi|x|/2)], & |x| \leq 2, \\
  0, & |x| > 2
\end{cases}
\]
which has one continuous derivative, and

\[ b(x) = \begin{cases} \exp[-1/(4 - |x|^2)], & |x| < 2, \\ 0, & |x| \geq 2 \end{cases} \]

which has infinitely many continuous derivatives, as may be shown by induction and l'Hôpital's rule.

### 1.5 Bump Functions on a Manifold

Suppose that \((M, T, A)\) is a \(d\)-dimensional differentiable manifold. This is a paracompact topological space, so every open cover has a locally finite refinement.

Given a chart \((G, \phi)\) and fixed \(x \in G\), it may be assumed WOLOG that

- \(\phi(x) = 0 \in \mathbb{R}^d\), else use \(\phi_x(z) \stackrel{\text{def}}{=} \phi(z) - \phi(x)\).
- \(\phi(G) \subset B(0, 2)\), else use \(\phi_{\epsilon}(z) \stackrel{\text{def}}{=} \epsilon \phi(z)\) with sufficiently small \(\epsilon > 0\).

Then, using a bump function \(b : \mathbb{R}^d \to \mathbb{R}\) from the previous section, define \(b_G : M \to \mathbb{R}\) by

\[ b_G(x) = \begin{cases} b \circ \phi(x), & x \in G \\ 0, & x \in M \setminus G \end{cases} \]

Each such function is differentiable on \(M\).

Now let \(M \subset \{G_\alpha : \alpha \in I\}\) be a locally finite cover. Define \(g : M \to \mathbb{R}\) by

\[ g(x) \stackrel{\text{def}}{=} \sum_{\alpha \in I} b_G(x), \quad x \in M, \]

which is finite and differentiable since each term is differentiable and, at each \(x\), there are only finitely many \(\alpha\) with \(b_G(x) > 0\) in the sum.

In addition, \(g(x) > 0\) since there is at least one strictly positive summand at each \(x \in M\).