Partitions of Unity.

Recall that a topological space $(X, \mathcal{T})$ is compact if every open cover of $X$ contains a finite subcover. This is a strong condition, but there are several related weaker conditions of interest.

1.1 Closures

Set $F \subset X$ is called closed iff its complement $F^c \overset{\text{def}}{=} X \setminus F$ is open.

The following are de Morgan’s formulas. Index set $I$ is arbitrary:

$$(\bigcap_{\alpha \in I} S_{\alpha})^c = \bigcup_{\alpha \in I} S_{\alpha}^c,$$

$$(\bigcup_{\alpha \in I} S_{\alpha})^c = \bigcap_{\alpha \in I} S_{\alpha}^c.$$

Use these to deduce that

- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

Define the closure $\bar{S}$ of a subset $S \subset X$ to be the smallest closed set that contains $S$, namely

$$\bar{S} = \bigcap_{\{F:F^c \in \mathcal{T}, S \subset F\}} F$$

which is the intersection of all the closed sets $F$ (whose complements are open sets $F^c \in \mathcal{T}$) that contain $S$.

Any closed subset $F$ of a compact set $K$ is compact: if $F \subset \bigcup_{\alpha \in I} G_{\alpha}$ is any open cover, then $K \subset F^c \cup \bigcup_{\alpha \in I} G_{\alpha}$ is an open cover of $K$, hence it has a finite subcover $F^c \cup G_1 \cup \cdots \cup G_N$, hence $F \subset G_1 \cup \cdots \cup G_N$ is a finite subcover of $F$.

Consequently, if $K$ is compact and $S \subset K$ also has $\bar{S} \subset K$, then $\bar{S}$ is compact.

A compact subset $K$ need not be closed. For example, let $X = \{a,b\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{a,b\}\}$. Then $\{a\}$ is compact but not closed, since its complement $\{a\}^c = \{b\} \notin \mathcal{T}$. (There are many other similarly contrived examples.)
However, any compact subset $K$ of a Hausdorff space must be closed. Prove this by showing that its complement $K^c$ is open, for which it suffices to show that each $x \in K^c$ belongs to some open set $G \subset K^c$. Find that $G$ as follows:

- For each $y \in K$, find disjoint open $G_y, H_y$ with $x \in G_y$ and $y \in H_y$. These exist by the Hausdorff property.
- Note that $K \subset \bigcup_{y \in K} H_y$ is an open cover.
- Since $K$ is compact, there exists a finite subcover which may be denoted $K \subset H_1 \cup \cdots \cup H_N$.
- Let $G_i$ be the open set around $x$ corresponding to $H_i$, $i = 1, \ldots, N$.
- Put $G \overset{\text{def}}{=} G_1 \cap \cdots \cap G_N$. This finite intersection of open sets is open.
- Note that $x \in G$.
- $G \subset K^c$, since $(\forall i) G \cap H_i = \emptyset$ and $K \subset \cup H_i$.

Conclude that $K^c$ is open, so $K$ is closed.

### 1.2 Local Compactness

A topological space $(X, T)$ is \textit{locally compact} if every point $x \in X$ has a compact neighborhood.

A $d$-dimensional manifold $\mathcal{M}$ is locally compact since it is a metric space that is locally homeomorphic to $\mathbb{E}^d$. Every $x \in \mathcal{M}$ belongs to some chart $(G, \phi)$ with open $G$, so there exists $\epsilon > 0$ such that $x \in B(x, 2\epsilon) \subset G$. But then

$$\bar{B}(x, \epsilon) \overset{\text{def}}{=} \{ y \in \mathcal{M} : d(x, y) \leq \epsilon \} \subset G,$$

and $\phi(\bar{B}(x, \epsilon)) \subset \phi(G) \subset \mathbb{E}^d$ is closed and bounded. Such sets are compact by the Heine-Borel theorem. Conclude that

$$\bar{B}(x, \epsilon) = \phi^{-1} (\phi(\bar{B}(x, \epsilon))) \subset \mathcal{M}$$

is a compact neighborhood of $x$.

### 1.3 Paracompactness

Start with the notion of \textit{refinement} of open covers: collection $\mathcal{G}' \subset \mathcal{T}$ is a refinement of $\mathcal{G} \subset \mathcal{T}$ iff

$$(\forall G \in \mathcal{G})(\exists G' \in \mathcal{G}') G' \subset G.$$  

For example, in a metric space $X$, $\mathcal{G}' = \{B(x, \epsilon/2) : x \in X\}$ is a refinement of $\mathcal{G} = \{B(x, \epsilon) : x \in X\}$.
Next, say that an open cover \( \{ G_\alpha : \alpha \in I \} \) is \textit{locally finite} iff every \( x \in X \) is contained in some neighborhood \( U_x \) that intersects only finitely many sets in the cover. Namely,

\[
(\forall x \in X) (\exists U_x \in \mathcal{T}) (x \in U_x \text{ and } \{ \alpha \in I : U_x \cap G_\alpha \neq \emptyset \} \text{ is finite}).
\]

Finally, say that \((X, \mathcal{T})\) is \textit{paracompact} iff every open cover has a locally finite refinement.

**Remark.** If there is a locally finite refinement \( G' = \{ G'_\beta : \beta \in J \} \) of \( G = \{ G_\alpha : \alpha \in I \} \), then there is an identically-indexed locally finite refinement \( \mathcal{H} = \{ H_\alpha : \alpha \in I \} \) of \( G \) such that \( H_\alpha \subset G_\alpha \) for each \( \alpha \in I \). It may be constructed by choosing a single \( \alpha = i(\beta) \in I \) for each \( \beta \in J \) such that \( G'_\beta \subset G_\alpha \), and then putting

\[
H_\alpha = \bigcup_{\{ \beta \in J : i(\beta) = \alpha \}} G'_\beta.
\]

It is clear that \( H_\alpha \subset G_\alpha \). But also, any neighborhood that intersects only finitely many \( G'_\beta \) can intersect at most finitely many \( H_\alpha \). (Since \( i \) is single-valued, there will be more intersecting \( \beta \) indices than \( \alpha \) indices.) Thus \( \mathcal{H} \) is a locally finite refinement of \( G \).

**Theorem.** A locally compact second countable Hausdorff space is paracompact.

**Proof.** See Proposition A1.6 in PartOfUnity_LocFiniteRefinements.pdf on the class website.

**Remark.** A differentiable manifold is locally compact (because it is finite dimensional), second countable (by definition), and Hausdorff (because it is a metric space). Hence any differentiable manifold is paracompact.

**Remark.** Any compact differentiable manifold is obviously paracompact since any finite cover is locally finite.

### 1.4 Bump functions

Start by constructing a continuously differentiable nonnegative function \( b : \mathbb{E}^d \to \mathbb{R} \) satisfying

- \(|x| \geq 2 \implies b(x) = 0\). Namely, \( b \) is supported in \( \overline{B}(0,2) \), the closed ball of radius 2 centered at \( 0 \).
- \(|x| \leq 1 \implies b(x) > 0\). Thus \( b \) is strictly positive in \( B(0,1) \).

Examples among elementary functions include

\[
b(x) = \begin{cases} 
\frac{1}{2}[1 + \cos(\pi |x|/2)], & |x| \leq 2, \\
0, & |x| > 2
\end{cases}
\]
which has one continuous derivative, and

\[ b(x) = \begin{cases} 
\exp\left[-1/(4 - |x|^2)\right], & |x| < 2, \\
0, & |x| \geq 2 
\end{cases} \]

which has infinitely many continuous derivatives, as may be shown by induction and l’Hôpital’s rule.

1.5 Bump Functions on a Manifold

Suppose that \((M, T, A)\) is a \(d\)-dimensional differentiable manifold. This is a paracompact topological space, so every open cover has a locally finite refinement.

Given a chart \((G, \phi)\) and fixed \(x \in G\), it may be assumed WOLOG that

- \(\phi(x) = 0 \in \mathbb{E}^d\), else use \(\phi_x(z) = \phi(z) - \phi(x)\).
- \(\phi(G) \subset B(0, 2)\), else use \(\phi_\epsilon(z) = \epsilon \phi(z)\) with sufficiently small \(\epsilon > 0\).

Then, using a bump function \(b : \mathbb{E}^d \to \mathbb{R}\) from the previous section, define \(b_G : M \to \mathbb{R}\) by

\[ b_G(x) = \begin{cases} 
b \circ \phi(x), & x \in G \\
0, & x \in M \setminus G 
\end{cases} \]

Each such function is differentiable on \(M\).

Now let \(M \subset \{G_\alpha : \alpha \in I\}\) be a locally finite cover. Define \(g : M \to \mathbb{R}\) by

\[ g(x) = \sum_{\alpha \in I} b_{G_\alpha}(x), \quad x \in M, \]

which is finite and differentiable since each term is differentiable and, at each \(x\), there are only finitely many \(\alpha\) with \(b_{G_\alpha}(x) > 0\) in the sum.

In addition, \(g(x) > 0\) since there is at least one strictly positive summand at each \(x \in M\).