1 Tangent Vectors, Spaces, and Bundles.

1.1 $T_x\mathcal{M}$ is a vector space

Fix $x \in \mathcal{M}$. Check three properties.

1.1.1 $T_x\mathcal{M}$ contains 0

The equivalence class of the constant curve $\gamma(t) \equiv x$, for which

$$\frac{d}{dt}(\phi \circ \gamma(t))\bigg|_{t=0} = 0 \in \mathbb{R}^d,$$

for any differentiable chart $(G, \phi)$ with $x \in G$, is the zero vector in $T_x\mathcal{M}$.

1.1.2 $v \in T_x\mathcal{M} \implies cv \in T_x\mathcal{M}$

Let $c \in \mathbb{R}$ be given. If $v \in T_x\mathcal{M}$ is represented by $\gamma$, then $\eta(t) = \gamma(ct)$ represents $cv$ since, for any differentiable chart $(G, \phi)$ with $x \in G$,

$$\frac{d}{dt}(\phi \circ \eta(t))\bigg|_{t=0} = \frac{d}{dt}(\phi \circ \gamma(ct))\bigg|_{t=0} = c \frac{d}{dt}(\phi \circ \gamma(t))\bigg|_{t=0},$$

by the chain rule. Note that $\eta$ is defined on the possibly smaller open interval $(-1/|c|, 1/|c|)$ rather than $(-1, 1)$. This technical problem may be overcome by using

$$\eta(t) = \gamma(tan^{-1}(ct))$$

whose domain is $(-\infty, \infty)$ and which gives the same chain rule result since

$$\frac{d}{dt}[tan^{-1}(ct)]\bigg|_{t=0} = c \frac{d}{dt}[tan^{-1}(t)]\bigg|_{t=0} = c \frac{1}{1+t^2}\bigg|_{t=0} = c$$

Alternatively, the chain rule may be avoided by representing $cv$ with the curve $\eta(t) \overset{\text{def}}{=} \phi^{-1}(c\phi \circ \gamma(t))$, which may be different for each coordinate map $\phi$:

$$\frac{d}{dt}[\phi \circ \eta(t)]\bigg|_{t=0} = \frac{d}{dt}[\phi \circ \phi^{-1}(c\phi \circ \gamma(t))]\bigg|_{t=0} = c \frac{d}{dt}[\phi \circ \gamma(t)]\bigg|_{t=0}.$$
1.1.3 $v, w \in T_x M \implies v + w \in T_x M$

Suppose that $v, w \in T_x M$ are equivalence classes of differentiable curves through $x$, represented respectively by $\gamma, \eta$. Let $(G, \phi)$ be any chart with $x \in G$. Then it is possible to define a “sum” of $\gamma, \eta$ as

$$\xi(t) \overset{\text{def}}{=} \phi^{-1}(\phi \circ \gamma(t) + \phi \circ \eta(t)),$$

since $\phi \circ \gamma$ and $\phi \circ \eta$ both belong to the linear space $E^d$.

WOLOG $\phi(x) = 0 \in E^d$, else replace it with the compatible map

$$\tilde{\phi}_x(z) \overset{\text{def}}{=} \phi(z) - \phi(x), \quad z \in G,$$

which has the same domain and differentiability but satisfies $\tilde{\phi}_x(x) = 0$.

Since homeomorphism $\phi$ is an open map, there exists $\epsilon > 0$ such that $B(0, 2\epsilon) \subset \phi(G)$. Since $\gamma, \eta$ are continuous, there exists $\delta > 0$ such that the small open interval $I = (-\delta, \delta) \subset (-1, 1)$ satisfies

$$\gamma(I) \subset G, \phi(\gamma(I)) \subset B(0, \epsilon); \quad \eta(I) \subset G, \phi(\eta(I)) \subset B(0, \epsilon);$$

Then the domain of $\xi$ includes $I$, since

$$t \in (-\delta, \delta) \implies \phi \circ \gamma(t) + \phi \circ \eta(t) \in B(0, 2\epsilon) \subset \phi(G).$$

Finally, reparametrize $\xi$ so that its domain includes $(-1, 1)$, again using the arctangent function:

$$\tilde{\xi}(t) \overset{\text{def}}{=} \xi(\delta \tan^{-1}(t/\delta)).$$

Then by the chain rule,

$$\frac{d}{dt}[\phi \circ \tilde{\xi}(t)]\bigg|_{t=0} = \frac{d}{dt}[\phi \circ \xi(t)]\bigg|_{t=0},$$

and thus

$$\frac{d}{dt}[\phi \circ \tilde{\xi}(t)]\bigg|_{t=0} = \frac{d}{dt}[\phi \circ \phi^{-1}(\phi \circ \gamma(t) + \phi \circ \eta(t))]\bigg|_{t=0} = \frac{d}{dt}[\phi \circ \gamma(t) + \phi \circ \eta(t)]\bigg|_{t=0} = \frac{d}{dt}[\phi \circ \gamma(t)]\bigg|_{t=0} + \frac{d}{dt}[\phi \circ \eta(t)]\bigg|_{t=0}. $$

Conclude that $\xi$ represents the equivalence class of $v + w$.

1.2 $d\phi(x)$ is a linear homeomorphism from $T_x M$ onto $E^d$.

1.2.1 $d\phi(x)$ is linear.

Let $u, v \in T_x M$ be given, represented by curves $\gamma, \eta$ through $x$, and suppose $c \in \mathbb{R}$ is also given.
$T_xM$ is a vector space, so $u + cv \in T_xM$ has a representative curve $\xi$ through $x$ which, except for domain adjustments, may be written as

$$\xi(t) = \phi^{-1}(\phi \circ \gamma(t) + c\phi \circ \eta(t)).$$

(Adjustments like $\phi \leftarrow \phi - \phi(x)$ and $t \leftarrow \delta \tan^{-1}(t/\delta)$ from Equations 2 and 3 would result in $\phi(x) = 0$ and domain $-1 < t < 1$ for all curves, without loss.) Then by definition,

$$d\phi(x)(u + cv) = d\xi \phi(x) = \frac{d\phi(\xi(t))}{dt} \bigg|_{t=0}$$

$$= d\left[\phi \circ \phi^{-1}(\phi \circ \gamma(t) + c\phi \circ \eta(t))\right] \bigg|_{t=0}$$

$$= d\left[\phi \circ \gamma(t)\right] \bigg|_{t=0} + cd \left[\phi \circ \eta(t)\right] \bigg|_{t=0}$$

$$= d\gamma \phi(x) + cd\eta \phi(x) = d\phi(x)(u) + cd\phi(x)(v).$$

Conclude that $d\phi(x)$ is linear.

1.2.2 $d\phi(x)$ is surjective.

Let $e = \{e_1, \ldots, e_d\}$ be the standard basis of $E^d$. Fix $k$ and parametrize a curve $\gamma$ through $x \in M$ with

$$\gamma(t) \overset{\text{def}}{=} \phi^{-1}(\phi(x) + te_k), \quad -1 < t < 1. \quad (4)$$

(If necessary to stay within $\phi(G)$ for all $-1 < t < 1$, replace $t \leftarrow \delta \tan^{-1}(t/\delta)$ using small enough $\delta > 0$.) Then the directional derivative of $\phi$ along $\gamma$ is

$$d\gamma \phi(x) = \frac{d}{dt} \left[\phi \circ \gamma(t)\right] \bigg|_{t=0} = \frac{d}{dt} \left[\phi \circ \gamma(t)\right] \bigg|_{t=0} = e_k.$$

Thus $\gamma$ represents a tangent vector $v_k$ for which $d\phi(x)(v_k) = d\gamma \phi(x) = e_k$.

Repeating the Eq.4 construction for all $k \in \{1, \ldots, d\}$ gives distinct tangent vectors $\{v_1, \ldots, v_d\} \subset T_xM$ with

$$d\phi(x)(v_k) = e_k, \quad k = 1, \ldots, d.$$

Now suppose that $p \in E^d$ is given. Write $p = \sum_k a_k e_k$, using the basis for $E^d$. Since $T_xM$ is a vector space, it contains the linear combination $w \overset{\text{def}}{=} \sum_k a_k v_k$.

Applying the linearity of $d\phi(x)$, compute

$$d\phi(x)(w) = d\phi(x) \left(\sum_k a_k v_k\right) = \sum_k a_k d\phi(x)(v_k) = \sum_k a_k e_k = p.$$

Hence $w \in T_xM$ is a preimage of $p$. Conclude that $d\phi(x)$ is surjective.
1.2.3 \( d\phi(x) \) is injective.

Since \( d\phi(x) \) is linear, it suffices to show that its nullspace is just \{0\}.

So suppose that \( u \in T_x\mathcal{M} \) satisfies \( d\phi(x)(u) = 0 \). Let \( \gamma \) be a curve through \( x \) that represent \( u \). Then by definition,

\[
0 = d\phi(x)(u) = d\gamma \circ \phi(x) = \frac{d}{dt} [\phi \circ \gamma(t)]_{t=0}.
\]

Now let \((H, \psi)\) be any chart in the maximal differentiable atlas for \( \mathcal{M} \) such that \( x \in G \cap H \). Let \( \tau = \psi \circ \phi^{-1} \) be the differentiable transition function on \( \phi(G \cap H) \), where \( \tau : \mathbb{E}^d \to \mathbb{E}^d \). Then \( D\tau(p) : \mathbb{E}^d \to \mathbb{E}^d \) is a \( d \times d \) matrix for any \( p \in \phi(G \cap H) \), and the chain rule may be used to evaluate:

\[
\frac{d}{dt} [\psi \circ \gamma(t)]_{t=0} = \frac{d}{dt} [\psi \circ \phi^{-1} \circ \phi \circ \gamma(t)]_{t=0} = \frac{d}{dt} [\tau \circ \phi \circ \gamma(t)]_{t=0} = D\tau(\phi(\gamma(0))) \frac{d}{dt} [\phi \circ \gamma(t)]_{t=0} = D\tau(\phi(x))0 = 0,
\]

since \( \gamma(0) = x \) and \( \phi(x) \in \phi(G \cap H) \). Hence \( u \) is the equivalence class of curves through \( x \) that give the zero vector as the directional derivative for every chart, namely the zero tangent vector.

1.2.4 Remarks on higher derivatives

Finding \( d\phi(x) \) consumes one derivative, which is all that is assumed to exist for a differentiable manifold. To define higher order derivatives, the atlas of charts on \( \mathcal{M} \) must contain coordinate functions with \( K > 1 \) derivatives (for \( C^K \) manifolds) or even infinitely many derivatives (for \( C^\infty \), or smooth manifolds).