Local Sine and Cosine Bases of Coifman and Meyer
and the Construction of Smooth Wavelets

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Abstract. We give a detailed account of the local cosine and sine bases of Coifman and Meyer. We describe some of their applications; in particular, based on an approach by Coifman and Meyer, we show how these local bases can be used to obtain arbitrarily smooth wavelets. The understanding of this material requires only a minimal knowledge of the Fourier transform and classical analysis. It is our intention to make this presentation accessible to all who are interested in Wavelets and their applications.

§1. Introduction.

It is often useful to focus on local properties of a signal. In precise mathematical language this means that if we are given a function \( f \) on \( \mathbb{R} \) and want to consider its properties on a finite interval \( I \), we can analyze the function multiplied by \( \chi_I \), the characteristic function of \( I \). We can, for example, form the Fourier series of \( \chi_I f \) with respect to a complete orthonormal system for \( L^2(I) \). An example of such a system is

\[
\left\{ \sqrt{\frac{2}{|I|}} \chi_I(x) \sin \frac{2k + 1}{2} \frac{\pi}{|I|} (x - \alpha) \right\},
\]

where \( \alpha \) is the left end point of \( I \), \( k = 0, 1, 2, \ldots \) (further discussion of this and other systems will be given in §3). If \( -\infty < \cdots < \alpha_j < \alpha_{j+1} < \cdots < \infty \), with \( \alpha_j \to \pm \infty \) as \( j \to \pm \infty \), and \( I_j = [\alpha_j, \alpha_{j+1}] \), we obtain an orthonormal system

\[
\varphi_{j,k}(x) = \sqrt{\frac{2}{|I_j|}} \chi_{I_j}(x) \sin \frac{2k + 1}{2} \frac{\pi}{|I_j|} (x - \alpha_j),
\]

where \( j \) ranges through the integers \( \mathbb{Z} \) and \( k \) through the non-negative integers \( \mathbb{Z}^+ \), that is a basis for \( L^2(\mathbb{R}) \). Expansions in terms of such bases are referred to
as “windowed” or “short time” Fourier transforms. Though such systems are appropriate for focusing on local properties (i.e. what happens on the interval $I_j$), the abrupt “cutoff” effected by the multiplication by the characteristic functions $\chi_{I_j}$ involve some undesirable artifacts. In [1], Coifman and Meyer introduced orthonormal bases of this general type that involve an arbitrarily smooth cut off. It is the purpose of this note to present the construction of such bases and, in addition, show some of their uses. In particular, we shall describe, in §4, how the smooth wavelets of Lemarié and Meyer [2] can be obtained from these bases. We hasten to add that this application was also pointed out to us by Coifman and Meyer.

Let us begin by trying to construct a projection $P_I$, given an interval $I$, that is similar to the one obtained by multiplication by $\chi_I$ but is “smoother”. Clearly, $P_I$ cannot have the form $f \to \rho_I f$, with $\rho = \rho_I$ a smooth function with support “close” to $I$, since the requirement that $P_I$ be idempotent forces $\rho$ to have values that are either 0 or 1. Perhaps a smooth version of $\chi_I$ can be “corrected” near the end points of $I$ so that we have a projection. In order to reduce the problem to only one end point, let us try to carry out this idea on the infinite ray $I = [0, \infty)$ and let $\rho$ be a smooth non-negative function supported on $[-\epsilon, \infty)$ such that $\rho(x) = 1$ if $x \geq \epsilon$. Let us also assume that $\rho$ shares with $\chi_I$ the property

$$\rho(x) + \rho(-x) = 1$$

for all $x \in \mathbb{R}$. In order to perform the above “correction” of the operator $f \to \rho f$ so as to obtain an orthogonal projection we pose the question: can we find a function $t$ such that

$$(P_I f)(x) = (P f)(x) \equiv \rho(x)f(x) + t(x)f(-x)$$

is an orthogonal projection? One immediate calculation shows that $P$ is self-adjoint if and only if $t(x) = t(-x)$. If, for the sake of simplicity, we assume $t$ is real-valued, this relation becomes, simply, that $t$ is an even function. The idempotent property $P^2 = P$, because of (1.3), becomes

$$\{\rho^2(x) + t(x)t(-x)\} f(x) + t(x) f(-x) = (P^2 f)(x) = (P f)(x) = \rho(x)f(x) + t(x)f(-x)$$

for all $f \in L^2(\mathbb{R})$. Again using (1.3) this equality holds if and only if $t(x)t(-x) = \rho(x)(1 - \rho(x)) = \rho(x)\rho(-x)$. Since $t$ is even this is equivalent to

$$t(x) = \pm \sqrt{\rho(x)\rho(-x)}.$$ 

This shows that, under these assumptions on $\rho$ and $t$, $P$ is a projection if and only if

$$(P f)(x) = \rho(x)f(x) \mp \sqrt{\rho(x)\rho(-x)}f(-x).$$

(1.4)
It is not hard to make an explicit construction of such functions \( \rho \) that are as smooth as desired: we begin by choosing an even non-negative function \( \psi \) with \( \text{supp } \psi \subset [-\epsilon, \epsilon] \) so normalized that

\[
\int_{\mathbb{R}} \psi = \frac{\pi}{2}
\]

Then let

\[
\theta(x) = \int_{-\infty}^{x} \psi(t)dt.
\]

An immediate consequence of the fact that \( \psi \) is even is that

\[
\theta(x) + \theta(-x) = \frac{\pi}{2}.
\]

(1.5)

We now put \( s_\epsilon(x) \equiv \sin \theta(x) \) and \( c_\epsilon(x) \equiv \cos \theta(x) \). It follows from (1.5) that \( c_\epsilon(x) = \cos \left( \frac{\pi}{2} - \theta(-x) \right) = s_\epsilon(-x) \). That is,

\[
c_\epsilon(x) = s_\epsilon(-x).
\]

(1.6)

Thus, the graph of \( c_\epsilon \) is the mirror image, through the vertical axis \( x = 0 \), of the graph of \( s_\epsilon \). We also have the relation

\[
s_\epsilon^2(x) + c_\epsilon^2(x) = 1.
\]

(1.7)

![The graphs of \( s_\epsilon \) and \( c_\epsilon \).](image)

If we now let \( \rho(x) = s_\epsilon^2(x) \) we see that \( \rho \) enjoys the above properties; in particular, (1.3) is an immediate consequence of (1.6) and (1.7). Equality (1.4) then becomes

\[
(Pf)(x) \equiv (P_0 f)(x) = s_\epsilon^2(x) f(x) \pm s_\epsilon(x) c_\epsilon(x) f(-x),
\]

(1.8)
where we use the notation $P_0$ to indicate that $P$ is associated with the ray $[0, \infty)$ (of course, $P_0$ also depends on $\varepsilon > 0$ and, in a few occasions, we show this additional dependence by writing $P_{0, \varepsilon}$ instead of $P_0$).

Had we chosen the half ray $(-\infty, 0]$ instead of $[0, \infty)$, completely analogous reasoning would lead us to the orthogonal projection $P^0$ given by the formula

$$ (P^0 f)(x) = c^2_e(x)f(x) \pm c_e(x)s_e(x)f(-x), \quad (1.9) $$

where, again, we do not explicitly denote the dependence of $P_0$ on $\varepsilon'$. We should also observe that, in each case, the projections also depend on the choice of sign before the second summand. Thus, we have introduced the four projections

$$ P^0_+ e', P^0_0, P^0_- e, P^0_{0, \varepsilon}. \quad (1.10) $$

§2. The construction of the projections $P_I$

associated with the interval $I = [\alpha, \beta]$.

We begin by translating $P_0$ and $P^0$ to any two points $\alpha, \beta$ in $\mathbb{R}$. Letting $t_\gamma$ be the translation operator defined by $(t_\gamma f)(x) = f(x - \gamma)$, we introduce the translates (by $\alpha$ and $\gamma$) of $P_0$ and $P^0$ by letting

$$ P_\alpha \equiv t_\alpha P_0 t_{-\alpha} \quad \text{and} \quad P^\beta \equiv t_\beta P^0 t_{-\beta}. \quad (2.1) $$

It is easy to check that

$$ (P_\alpha f)(x) = s^2_e(x - \alpha)f(x) \pm s_e(x - \alpha)c_e(x - \alpha)f(2\alpha - x), \quad (2.2) $$

$$ (P^\beta f)(x) = c^2_e(x - \beta)f(x) \pm c_e(x - \beta)s_e(x - \beta)f(2\beta - x). $$

Since $t^*_\gamma = t_{-\gamma} = t^{-1}_\gamma$, we see immediately that $P_\alpha$ and $P^\beta$ are orthogonal projections for each $\alpha$ and $\beta$. Observe that $x$ and $2\gamma - x$ are symmetric with respect to the line $x = \gamma$ (they lie on opposite sides of $\gamma$ and at a distance $|x - \gamma|$ from $\gamma$). We say that a function $g$ is even with respect to $\gamma$ if $g(2\gamma - x) = g(x)$ for all $x$. It is an immediate consequence of (2.2) that $gP_\alpha = P_\alpha g$ (and $gP^\beta = P^\beta g$) when $g$ is even with respect to $\alpha(\beta)$. Using this commutativity with $g = \chi_{[\alpha - \varepsilon, \alpha + \varepsilon]}$ and $g = \chi_{[\beta - \varepsilon', \beta + \varepsilon']}$, the properties of $s_e, c_e, s'_e, c'_e$, and (2.2) we see that

$$ P_\alpha P^\beta f = \chi_{[\alpha - \varepsilon, \alpha + \varepsilon]} P_\alpha f + \chi_{[\alpha + \varepsilon, \beta - \varepsilon'] f} + \chi_{[\beta - \varepsilon', \beta + \varepsilon']} P^\beta f = P^\beta P_\alpha f \quad (2.3) $$

as long as $\alpha + \varepsilon \leq \beta - \varepsilon'$. This allows us to define $P_I = P[\alpha, \beta]$ by letting

$$ P_{[\alpha, \beta]} \equiv P_\alpha P^\beta = P^\beta P_\alpha \quad (2.4) $$

whenever $-\infty < \alpha < \beta < \infty$. Because of this commuting property it is clear that $P_I$ is an orthogonal projection. In view of (1.10) we remark that $P_I$ depends on the choices of $+$ and $-$ in $P_\alpha, P^\beta$ and on $\varepsilon, \varepsilon'$ (as long as $\alpha + \varepsilon \leq \beta - \varepsilon'$).
We shall discuss the importance of this dependency a little later. Before doing this we introduce the bell over I. This is the function $b_I$ that depends on $\alpha, \beta, \epsilon, \epsilon'$, but not on the choice of signs, defined by

$$b_I(x) \equiv s_\epsilon(x - \alpha)c_{\epsilon'}(x - \beta)$$  \hspace{1cm} (2.5)

for all $x \in \mathbb{R}$. We list the basic properties of this bell function; each is an easy consequence of (2.5) and the properties of the functions $s_\epsilon, c_\epsilon, s_{\epsilon'}, c_{\epsilon'}$ developed in §1 (in particular (1.6) and (1.7)):

The function $b_I$ satisfies

1. Supp $b_I = [\alpha - \epsilon, \beta + \epsilon']$.  
2. $b_I(x) = s_\epsilon(x - \alpha)$;  
3. $b_I(2\alpha - x) = s_\epsilon(\alpha - x) = c_\epsilon(x - \alpha)$;  
4. $\beta^2_I(x) + b_I^2(2\alpha - x) = 1$.  
5. Supp $b_I(x)b_I(2\alpha - x) = [\alpha - \epsilon, \alpha + \epsilon]$.  
6. $b_I(x) = 1$ when $x \in [\alpha + \epsilon, \beta - \epsilon']$.  

On $[\beta - \epsilon', \beta + \epsilon']$:

7. $b_I(x) = c_{\epsilon'}(x - \beta)$;  
8. $b_I(2\beta - x) = c_{\epsilon'}(\beta - x) = s_{\epsilon'}(x - \beta)$;  
9. $\beta^2_I(x) + b_I^2(2\beta - x) = 1$.  
10. Supp $b_I(x)b_I(2\beta - x) = [\beta - \epsilon', \beta + \epsilon']$.  
11. When $x \in \text{Supp} b_I = [\alpha - \epsilon, \beta + \epsilon']$

$$\beta^2_I(x) + b_I^2(2\beta - x) + b_I^2(2\alpha - x) = 1.$$

The bell $b$ over $I = [\alpha, \beta]$.

The last property, obviously an immediate consequence of (4), (6) and (9), is perhaps, the most useful. Observe that the most important feature of these properties is the focus on the behaviour of $b_I$ on the three intervals $[\alpha - \epsilon, \alpha + \epsilon], [\alpha + \epsilon, \beta - \epsilon']$ and $[\beta - \epsilon', \beta + \epsilon']$.

The projection $P_I = P_{[\alpha, \beta]}$ has a simple expression in terms of the bell function $b_I$:

$$(P_I f)(x) = b_I^2(x)f(x) + b_I(x)b_I(2\alpha - x)f(2\alpha - x) \pm b_I(x)b_I(2\beta - x)f(2\beta - x).$$  \hspace{1cm} (2.7)
This formula is an immediate consequence of (2.3) and (2.6). It exhibits very clearly the dependence of \( P_I \) on the choice of signs associated with the endpoints \( \alpha \) and \( \beta \) of the interval \( I \). When \( \epsilon \) and \( \epsilon' \) are fixed we are dealing with four projections that are dependent on the two polarities (that is, the choice of signs) at each endpoint. The polarities are particularly important when we want to study the properties of \( P_I \) and \( P_J \) when \( I \) and \( J \) are adjacent intervals. The dependence of these projections on the choice of \( \epsilon \) and \( \epsilon' \) is also important in these considerations. Let us examine this in detail.

Two adjacent intervals \( I = [\alpha, \beta] \) and \( J = [\beta, \gamma] \) are compatible and have bells that are compatible if \( \alpha - \epsilon < \alpha < \alpha + \epsilon \leq \beta - \epsilon' < \beta < \beta + \epsilon' \leq \gamma - \epsilon'' < \gamma < \gamma + \epsilon'' \) and

\[
b_I(x) = s_\epsilon(x - \alpha)c_{\epsilon'}(x - \beta), \quad b_J(x) = s_{\epsilon'}(x - \beta)c_{\epsilon''}(x - \gamma).
\]

Clearly if we apply (2.6)(3) to \( b_I \) (with \( J, \beta, \epsilon \) replacing \( I, \alpha, \epsilon \)) we have \( b_I(2\beta - x) = c_{\epsilon'}(x - \beta) \) when \( x \in [\beta - \epsilon', \beta + \epsilon'] \); by (2.6)(7), on the other hand, \( b_I(x) = c_{\epsilon'}(x - \beta) \) when \( x \in [\beta - \epsilon', \beta + \epsilon'] \) (assuming \( I \) and \( J \) are compatible). Thus, when \( I \) and \( J \) are compatible,

\[
b_I(x) = b_J(2\beta - x)
\]

if \( x \in [\beta - \epsilon', \beta + \epsilon'] \). A similar use of (2.7)(6) and (9) gives us

\[
b_I^2(x) + b_J^2(x) = 1
\]

when \( x \in [\alpha + \epsilon, \gamma - \epsilon''] \). This last relation extends to the equality

\[
\sqrt{b_I^2(x) + b_J^2(x)} = s_\epsilon(x - \alpha)c_{\epsilon''}(x - \gamma)
\]

for all \( x \), which is equivalent to

\[
b_I^2 + b_J^2 = b_{IJ}^2
\]

whenever \( I \) and \( J \) are compatible adjacent intervals. As mentioned before, the bell functions \( b_I \) are independent of the choices of sign associated with the endpoints of \( I \). The polarity of two adjacent intervals \( I \) and \( J \), however, plays an important role if we desire that the projections \( P_I \) and \( P_J \) satisfy an additive property analogous to (2.10). More precisely we have the following result:

**Proposition (2.11).** Suppose \( I = [\alpha, \beta] \) and \( J = [\beta, \gamma] \) are adjacent compatible intervals and \( P_I, P_J \) have opposite polarity at \( \beta \). Then \( P_I + P_J \) is the orthogonal projection \( P_{IJ} \):

\[
P_I + P_J = P_{IJ}.
\]

Moreover, \( P_I \) and \( P_J \) are orthogonal to each other:

\[
P_IP_J = P_JP_I = 0.
\]
Local Sine and Cosine Bases

Proof: Equality (2.12) is an immediate consequence of (2.7), (2.8), and (2.10) (since the terms involving the end point $\beta$ cancel each other). Equality (2.13) is a consequence of the general result in Hilbert space theory: if $P$ and $Q$ are orthogonal projections such that $P + Q$ is an orthogonal projection, then $PQ = 0$. Here is a simple proof of this fact. If $(P + Q)^2 = P + Q$, the idempotent properties of $P$ and $Q$ give us $PQ = -QP$. Thus, $PQ = P^2Q = -PQP = QP^2 = QP$. Since $PQ = -QP$ and $PQ = QP$ it follows that $PQ = QP = 0$.

Another consequence of formula (2.7) is a simple characterization of the image $\mathcal{H}_I = P_I L^2(\mathbb{R}^n)$ of $P_I$. Let us say that a function $f$ is even (odd) with respect to $\alpha$ on $[\alpha - \epsilon, \alpha + \epsilon]$ if and only if $f(2\alpha - x) = f(x)$ ($f(2\alpha - x) = -f(x)$) when $x \in [\alpha - \epsilon, \alpha + \epsilon]$. Observe that (2.7) can be written in the form

$$(P_I f)(x) = b_I(x) S(x),$$

where $S(x) = b_I(x)f(x) \pm b_I(2\alpha - x)f(2\alpha - x) \pm b_I(2\beta - x)f(2\beta - x)$. But this function is odd (even) with respect to $\alpha$ on $[\alpha - \epsilon, \alpha + \epsilon]$ if $- (+)$ is chosen before the second term in (2.7) and is odd (even) with respect to $\beta$ on $[\beta - \epsilon', \beta + \epsilon']$ if $- (+)$ is chosen before the third term. If this odd/even property corresponds, at an end point of $I$, with the $-/+$ polarity of $I$ at this end point, we say that $f$ has the same polarity as $I$ has at this point. Our characterization of $\mathcal{H}_I$, then, is given by:

**Theorem (2.15).** $f \in \mathcal{H}_I = P_I L^2(\mathbb{R})$ if and only if $f = bS$, where $S$ is a function in $L^2(\mathbb{R})$ having the same polarity as $I$ at its end points.

Proof: From (2.14) we see that each element of $\mathcal{H}_I$ has the form $f = bS$. Now suppose $f$ has this form. Apply (2.7) to $f = bS$ and we obtain:

$$(P_I bS)(x) = b^2(x)b(x)S(x) \pm b(x)b^2(2\alpha - x)S(2\alpha - x) \pm b(x)b^2(2\beta - x)S(2\beta - x)$$

$$= \chi_{[\alpha-\epsilon,\alpha+\epsilon]}(x)b(x)S(x)\{b^2(x) + b^2(2\alpha - x)\} + \chi_{[\alpha+\epsilon,\beta-\epsilon']} (x)b(x)S(x) +$$

$$\chi_{[\beta-\epsilon',\beta+\epsilon']} (x)b(x)S(x)\{b^2(x) + b^2(2\beta - x)\} = b(x)S(x)$$

by (2.6) (4), (5), (6), (9), and (10).

Finally, let us show how to use these projections to decompose $L^2(\mathbb{R})$ into a direct sum of mutually orthogonal subspaces that are images under such projections:

$$L^2(\mathbb{R}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k.$$  

We do this as follows: choose a sequence $\{\alpha_k\}_{k \in \mathbb{Z}}$ of reals and accompanying positive numbers $\{\epsilon_k\}$ such that

$$\alpha_k + \epsilon_k < \alpha_{k+1} - \epsilon_{k+1}$$
for all $k$. Thus, each pair of adjacent intervals $I_{k-1} = [\alpha_{k-1}, \alpha_k]$ and $I_k = [\alpha_k, \alpha_{k+1}]$ is a compatible pair. Let $b_k = b_{I_k}$ be the bell over $I_k$ and $P_k \equiv P_{I_k}$. Let us also choose these projections so that they have opposite polarity at $\alpha_k$.

We then have $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} I_k$ if $\lim_{k \to \pm \infty} \alpha_k = \pm \infty$. Since

$$\bigcup_{k=-N}^{N} I_k = [\alpha_{-N}, \alpha_{N+1}],$$

it follows from (2.12) that

$$\sum_{k=-N}^{N} P_k = P_{[\alpha_{-N}, \alpha_{N+1}]}$$

(2.17)

Letting $\mathcal{H}_k \equiv P_k L^2(\mathbb{R})$, (2.13) assures us that $\mathcal{H}_k \perp \mathcal{H}_l$ if $k \neq l$. From (2.3) we see that

$$P_{[-\alpha_{N}, \alpha_{N+1}]} f = \chi_{[\alpha_{-N} + \epsilon_{-N}, \alpha_{N+1} + \epsilon_{N+1}]} f + E_N f,$$

where $E_N f$ is a function supported in the two intervals

$$[\alpha_{-N} - \epsilon_{-N}, \alpha_{-N} + \epsilon_{-N}] \text{ and } [\alpha_{N+1} - \epsilon_{N+1}, \alpha_{N+1} + \epsilon_{N+1}].$$

On the first interval $|(E_N f)(x)|$ is dominated by $|f(x)| + |f(2\alpha_{-N} - x)|$ and, on the second, by $|f(x)| + |f(2\alpha_{N+1} - x)|$. Hence, $\|E_N f\|_2 \to 0$ as $N \to \infty$ and it follows that

$$\lim_{N \to \infty} \|f - P_{[-\alpha_{N}, \alpha_{N+1}]} f\|_2 = 0.$$

These considerations clearly give us the decomposition (2.16).

§3. The local cosine and sine bases for $L^2(\mathbb{R})$.

Let us fix an interval $I$ and consider the problem of constructing “natural” orthonormal bases for the spaces $\mathcal{H}_I = P_I L^2(\mathbb{R})$. There are four such spaces if we take into account the two possible polarities at each end point of $I$. We ask, first, the simpler question: if the projection is the one obtained by multiplication by $\chi_I$, what are the “natural” bases of the image space $L^2(I)$ from the point of view of a harmonic analyst? Simplifying further, let us assume $I = [0, 1]$. Motivated by the polarity properties we have been discussing, we seek some orthonormal bases of $L^2(0, 1)$ that reflect these properties. Given $f \in L^2(0, 1)$ let us extend it to the interval $[0, 2]$ so that it gives us a function $\tilde{f}$ that is even with respect to 1 on this larger interval. Analytically this means $\tilde{f}(x) = f(2 - x)$ for $x \in [1, 2]$. We then extend $\tilde{f}$ to an odd function on $[-2, 2]$. 
The extension of \( f \) on \([0,1]\) to \([-2,2]\) so that it is even at 1 and odd at 0.

This last function can then be developed into a Fourier series on \([-2,2]\] by means of the orthonormal basis

\[
\{ \frac{1}{2}, \frac{1}{\sqrt{2}} \sin \frac{k \pi x}{2}, \frac{1}{\sqrt{2}} \cos \frac{k \pi x}{2} \}, k = 1, 2, \ldots
\]  

(3.1)

Since we are dealing with an odd function on \([-2,2]\], the even part \(\{ \frac{1}{2}, \frac{1}{\sqrt{2}} \cos \frac{k \pi x}{2} \}, k = 1, 2, \ldots\), plays no rôle in this expansion. For the same reason, among the remaining terms, only those that are even with respect to 1 on \([0,2]\] give us (possibly) non-zero coefficients in this Fourier series development. This shows that \( f \) can be expanded on \([0,1]\) in terms of the orthogonal family \(\{ \sin \frac{2k+1}{2} \pi x \}, k = 0, 1, 2, \ldots\)

Had we extended \( f \) to obtain the other three pairs of polarities at the points 0 and 1, we would have obtained three other subcollections of the family (3.1). These considerations give us

**Proposition (3.2).** Each of the following four systems forms an orthonormal basis for \(L^2(0,1)\):

(i) \(\{ \sqrt{2} \sin \frac{2k+1}{2} \pi x \}, k = 0, 1, 2, \ldots\);

(ii) \(\{ \sqrt{2} \sin k \pi x \}, k = 1, 2, 3, \ldots\);

(iii) \(\{ \sqrt{2} \cos \frac{2k+1}{2} \pi x \}, k = 0, 1, 2, \ldots\);

(iv) \(\{ 1, \sqrt{2} \cos k \pi x \}, k = 1, 2, 3, \ldots\).

The polarities of each of the functions in the first basis are \((-\,\,+,\,\,+\,\,\,+)\) at \((0,1)\), in the second basis they are \((-\,\,-\,\,+\,\,-\,\,+)\), in the third they are \((+,\,\,-\,\,\,\,+)\) and in the fourth they are \((+,\,\,+)\).

Let us now return to the study of the space \( \mathcal{H}_I = \mathcal{H}_{[0,1]} = P_I L^2(\mathbb{R}) \). To fix our ideas let us assume \( P_I \) is chosen with negative polarity at 0 and positive polarity at 1. The bell over \( I = [0,1] \) in this case is \( b(x) = b_I(x) = s_\epsilon(x) c_{\epsilon'}(x-1) \) and \( P = P_I \) is given (see (2.7)) by

\[
(P f)(x) = b(x) \{ b(x) f(x) - b(-x) f(-x) + b(2-x) f(2-x) \} \equiv b(x) S(x). \quad (3.3)
\]

If we restrict \( S \) to \([0,1]\), using the first basis in Proposition (3.2), we have

\[
S(x) = \sqrt{2} \sum_{k=0}^{\infty} c_k \sin \frac{2k+1}{2} \pi x, \quad (3.4)
\]
where the equality may be interpreted in the norm of $L^2(I)$, and the coefficients $c_k$ are given by the equality

$$c_k = \sqrt{2} \int_0^1 S(x) \sin \frac{2k + 1}{2} \pi x \, dx.$$  \hspace{1cm} (3.5)

But each of the functions $\sin \frac{2k+1}{2} \pi x$ satisfy the same polarity properties as $S$. It follows that equality (3.4) is valid on $[-\epsilon, 1 + \epsilon']$ and the convergence can be taken in the norm of $L^2(-\epsilon, 1 + \epsilon')$. Multiplying this new equality on both sides by $b(x)$ we see that any $f \in \mathcal{H}_I$ satisfies

$$f(x) = \sqrt{2} \sum_{k=0}^{\infty} c_k b(x) \sin \frac{2k + 1}{2} \pi x$$ \hspace{1cm} (3.6)

in $L^2(-\epsilon, 1 + \epsilon')$.

We claim the system $\{\sqrt{2}b(x) \sin \frac{2k+1}{2} \pi x, k = 0, 1, 2, \ldots\}$, is an orthonormal basis of $\mathcal{H}_I$ and $\{c_k\}$, given by (3.5), is the sequence of coefficients of $f \in \mathcal{H}_I$ with respect to this basis. It is clear from our discussion that in order to establish this claim all we need to show is that this system is orthonormal in $\mathcal{H}_I$. This is done by a simple calculation in which (2.6) and (2.7) play an important rôle: Let $f(x) = \sqrt{2}b(x) \sin \frac{2l+1}{2} \pi x$, then, by two changes of variables,

$$\int_{-\epsilon}^{\epsilon} f(x) \sqrt{2}b(x) \sin \frac{2k + 1}{2} \pi x \, dx =$$

$$\sqrt{2} \int_{0}^{\epsilon} \{f(x)b(x) - f(-x)b(-x)\} \sin \frac{2k + 1}{2} \pi x \, dx =$$

$$2 \int_{0}^{\epsilon} \{s^2(x) + c^2(x)\} \frac{2l + 1}{2} \pi x \left(\sin \frac{2k + 1}{2} \pi x\right) \, dx =$$

$$2 \int_{0}^{\epsilon} \left(\sin \frac{2l + 1}{2} \pi x\right) \left(\sin \frac{2k + 1}{2} \pi x\right) \, dx$$

and

$$\int_{1-\epsilon'}^{1+\epsilon'} f(x) \sqrt{2}b(x) \sin \frac{2k + 1}{2} \pi x \, dx =$$

$$\sqrt{2} \int_{1-\epsilon'}^{1} \{f(2-x)b(2-x) + f(x)b(x)\} \sin \frac{2k + 1}{2} \pi x \, dx =$$

$$2 \int_{1-\epsilon'}^{1} \{s^2(x-1) + c^2(x-1)\} \frac{2l + 1}{2} \pi x \left(\sin \frac{2k + 1}{2} \pi x\right) \, dx =$$

$$2 \int_{1-\epsilon'}^{1} \left(\sin \frac{2l + 1}{2} \pi x\right) \left(\sin \frac{2k + 1}{2} \pi x\right) \, dx.$$
\[
\int_{-\epsilon}^{1+\epsilon} f(x) \sqrt{2b(x)} \sin \left( \frac{2k + 1}{2} \pi x \right) dx = 2 \int_{0}^{1} \left( \sin \left( \frac{2l + 1}{2} \pi x \right) \sin \left( \frac{2k + 1}{2} \pi x \right) \right) dx = \delta_{kl}
\]

the last equality being a consequence of (3.2)(i).

Had we chosen the other polarities, \((-,-), (+,-), (+,+),\) in the definitions of \(P = P_{[0,1]}\), we reach the same conclusions if, instead of the system (3.2)(i), we use the systems (ii), (iii), and (iv) respectively. The case of the general interval \(I = [\alpha, \beta]\) now follows from these results by simple translation and dilation arguments. More precisely, we obtain

**Theorem (3.7).** If \(P_I = P_{[\alpha,\beta]}\) is the projection associated with negative polarity at \(\alpha\) and positive polarity at \(\beta\), then the system

\[
(i) \left\{ \sqrt{\frac{2}{|I|}} b_I(x) \sin \frac{2k + 1}{2} \frac{\pi}{|I|} (x - \alpha) \right\}, k = 0, 1, 2, \ldots, \tag{3.8}
\]

is an orthonormal basis for \(\mathcal{H}_I = P_I L^2(\mathbb{R})\). If we choose the polarities \((-,-), (+,-),\) and \((+,+)\) at \((\alpha, \beta)\), the same result is true if we use (respectively) the systems

\[
(ii) \left\{ \sqrt{\frac{2}{|I|}} b_I(x) \sin k \frac{\pi}{|I|} (x - \alpha) \right\}, k = 1, 2, 3, \ldots;
\]

\[
(iii) \left\{ \sqrt{\frac{2}{|I|}} b_I(x) \cos \frac{2k + 1}{2} \frac{\pi}{|I|} (x - \alpha) \right\}, k = 0, 1, 2, \ldots; \tag{3.8}
\]

\[
(iv) \left\{ \frac{1}{\sqrt{|I|}} b_I(x), \sqrt{\frac{2}{|I|}} b_I(x) \cos k \frac{\pi}{|I|} (x - \alpha) \right\}, k = 1, 2, 3, \ldots,
\]

instead of (3.8)(i).

We can now combine the results obtained at the end of §2 with this theorem to obtain the local sine and cosine bases of Coifman and Meyer. Let a sequence \(\{\alpha_j\}\) be selected as described at the end of §2. Thus, \(\alpha_j < \alpha_{j+1}, \lim_{j \to \pm \infty} \alpha_j = \pm \infty\) and there exists an accompanying sequence \(\{\epsilon_j\}\) such that

\[
\alpha_j + \epsilon_j \leq \alpha_{j+1} - \epsilon_{j+1}
\]

for all \(j \in \mathbb{Z}\). Let \(P_j = P_{[\alpha_j, \alpha_{j+1}]}\) be constructed with, say, negative polarity at \(\alpha_j\) and positive polarity at \(\alpha_{j+1}\). Let

\[
b_{jk}(x) = b_{[\alpha_j, \alpha_{j+1}]}(x) \sqrt{\frac{2}{\alpha_{j+1} - \alpha_j}} \sin \frac{2k + 1}{2} \frac{\pi}{\alpha_{j+1} - \alpha_j} (x - \alpha_j), \tag{3.9}
\]
$j \in \mathbb{Z}, k = 0, 1, 2, \ldots$. Our discussion (in particular (2.16) and (3.7)) allows us to conclude that the functions in (3.9) form an orthonormal basis for $L^2(\mathbb{R})$. We can change the polarity for each projection $P_j$ and arrive at the same conclusion as long as the polarity for $P_{j-1}$ at $\alpha_j$ is opposite to that of $P_j$ at $\alpha_j$; moreover, each such choice must be accompanied by exchanging the basis (3.9) with the basis in Theorem (3.7) that corresponds to these choices in polarity.

Here are some further observations that follow easily from the results we have obtained:

**Theorem (3.10).** Suppose $f \in \mathcal{H}_I$, where $I = [\alpha, \beta]$ is a finite interval in $\mathbb{R}$ and the polarities are, say, $-+$ at $\alpha$ and $\beta$, then the series

$$
\sqrt{\frac{2}{|I|}} \sum_{k=0}^{\infty} c_k b(x) \sin \frac{2k + 1}{2} \frac{\pi}{|I|} (x - \alpha)
$$

converges to $f(x)$ a.e. in $[\alpha - \epsilon, \beta + \epsilon']$. If $g \in L^2(\mathbb{R})$, then the development of $g$ in terms of the basis (3.9) converges a.e. to $g(x)$.

**Proof:** If $f = b_1 S_I = b S \in \mathcal{H}_I$ we have shown that,

$$
f(x) = \sum_{k=0}^{\infty} c_k b(x) \sqrt{\frac{2}{|I|}} \sin \frac{2k + 1}{2} \frac{\pi}{|I|} (x - \alpha)
$$

with convergence in the $L^2(I)$ norm. But we observed that the coefficients

$$
c_k = \sqrt{\frac{2}{|I|}} \int_{\alpha - \epsilon}^{\beta + \epsilon'} f(x) b(x) \sin \frac{2k + 1}{2} \frac{\pi}{|I|} (x - \alpha) dx,
$$

$k = 0, 1, 2, \ldots$, can also be calculated as the sine series coefficients

$$
c_k = \sqrt{\frac{2}{|I|}} \int_{\alpha}^{\beta} S(x) \sin \frac{2k + 1}{2} \frac{\pi}{|I|} (x - \alpha) dx.
$$

This is a consequence of (3.5) (after an appropriate translation and dilation).

Since $f \in L^2(\mathbb{R}), S = b^{-1} f$ is square integrable over $[\alpha, \beta]$. It then follows from Carleson’s theorem that the series

$$
\sqrt{\frac{2}{|I|}} \sum_{k=0}^{\infty} c_k \sin \frac{2k + 1}{2} \frac{\pi}{|I|} (x - \alpha) = S(x)
$$

converges to $S(x)$ a.e. in $[\alpha, \beta]$. Multiplying both sides by $b(x)$ we then have the validity of (3.11) for a.e. $x \in [\alpha, \beta]$. Since both sides are odd with respect to $\alpha$ on $[\alpha - \epsilon, \alpha + \epsilon]$ and even with respect to $\beta$ on $[\beta - \epsilon', \beta + \epsilon']$, this a.e. convergence extends to $[\alpha - \epsilon, \beta + \epsilon']$.

$\blacksquare$
Clearly there are versions of Theorem (3.10) for each choice of polarities associated with the interval \([\alpha, \beta]\). If the sequence \(\{\alpha_j\}\) (and the accompanying sequence \(\{\epsilon_j\}\)) gives us a family of mutually orthogonal projections \(P_j\) whose ranges span \(L^2(0, \infty)\) (by choosing \(\alpha_j \to 0\) as \(j \to -\infty\) and \(\alpha_j \to \infty\) as \(j \to \infty\)), we obtain an orthonormal basis of those functions in \(L^2(\mathbb{R})\) that are the Fourier transforms of the elements of the classical Hardy space \(H^2\). In the next section we describe another application in which it is useful to think of these local bases as the Fourier transform images of interesting bases of \(L^2(\mathbb{R})\).

We make one final comment before presenting the application that was just mentioned. The projections \(P_I\) are, indeed, “smoother” than the ones obtained by multiplication by \(\chi_t\). They are not, however, necessarily “smoothing.” For example, let \(f = \chi_{[0,1]}\) and \(P_I = P_{[0,1]}\). If \(I\) has negative polarity at 0, \(P_I f\) still has a jump at 0. Positive polarity at 0 on the other hand, “smooths out” \(f\) near 0.


Lemarié and Meyer [2] constructed a wavelet basis

\[
\{w_{k,n}(x)\} = \{2^{-k/2}w(2^{-k}x - n)\},
\]

\(k, n \in \mathbb{Z}\), where the “mother function” \(w\) belongs to \(\mathcal{S}(\mathbb{R})\) (in fact, it is the restriction of an entire function on \(\mathbb{C}\)) and

\[
\text{Supp } \hat{w} \subset [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \frac{8\pi}{3}].
\]

This means, in particular, that \(\{w_{k,n}\}\) is an orthonormal basis for \(L^2(\mathbb{R})\). This basis furnishes us with an example of a smooth “Multi Resolution Analysis” as described in [Ma] and [M]. We show that this construction can be carried out in an easy and natural way by using the local bases we have just presented.

Let \(I = [\pi, 2\pi], \epsilon = \frac{\pi}{3}, \epsilon' = 2\epsilon = \frac{2\pi}{3}\). Consider the orthogonal projection \(P_I\) with polarity (\(+,-\)). The bell function \(b = b_I\) associated with \(P_I\) will not have an interval of constancy since \(\pi + \epsilon = 2\pi - \epsilon'\). Its construction is explicitly given by (2.5) with \(\alpha = \pi, \beta = 2\pi, \epsilon = \frac{\pi}{3}, \epsilon' = \frac{2\pi}{3}\). The range of this projection is the subspace generated by the orthonormal basis

\[
\psi(n; \xi) = \sqrt{\frac{2}{\pi}} b(\xi) \cos \frac{2n + 1}{2} (\xi - \pi),
\]

\(n = 0, 1, 2, \ldots\) (see Theorem (3.7) and, in particular, (3.8) (iii)). The dilations by \(2^k, k \in \mathbb{Z}\), then give us the projections \(P_k = P_{[2^{-k}\pi, 2^{1-k}\pi]}\) with ranges spanned by the orthonormal basis

\[
\psi_k(n; \xi) = 2^{k/2} \psi(n; 2^k \xi),
\]

\(k \in \mathbb{Z}\). It follows from the material in §3. (see, in particular, the observation following Theorem (3.10), concerning the spanning of \(L^2(0, \infty)\)), that, with

Local Sine and Cosine Bases

13
this choice of polarity, the collection defined by (4.3) forms an orthonormal basis of $L^2(0, \infty)$.

Let us also carry out the completely analogous construction based on the local sine basis (3.8) (i):

$$\varphi(n; \xi) = \sqrt{\frac{2}{\pi}} b(\xi) \sin \frac{2n + 1}{2} (\xi - \pi),$$

$n = 0, 1, 2, \ldots$. We then obtain the orthonormal basis of $L^2(0, \infty)$

$$\varphi_k(n; \xi) = 2^{k/2} \varphi(n; 2^k \xi), \quad (4.4)$$

$k \in \mathbb{Z}$.

In order to obtain an orthonormal basis for $L^2(\mathbb{R})$ we consider the even extensions of the functions (4.3) and the odd extensions of the functions (4.4) to all of $\mathbb{R}$. More precisely, let

$$\Psi_{k,n}(\xi) \equiv \frac{1}{\sqrt{2\pi}} 2^{k/2} b(2^k |\xi|) \cos \frac{2n + 1}{2} (2^k |\xi| - \pi)$$

and

$$\Phi_{k,n}(\xi) \equiv \frac{1}{\sqrt{2\pi}} 2^{k/2} b(2^k |\xi|) (\text{sgn}\xi) \sin \frac{2n + 1}{2} (2^k |\xi| - \pi).$$

**Theorem (4.5).** The collection of functions

$$\alpha_{k,n} \equiv \Psi_{k,n} + i\Phi_{k,n} \quad \text{and} \quad \beta_{k,n} = \Phi_{k,n} + i\Psi_{k,n},$$

$k = 0, \pm 1, \pm 2, \ldots, n = 0, 1, 2, \ldots$, is an orthonormal basis for $L^2(\mathbb{R})$.

**Proof:** Let $\langle f, g \rangle = \int_{\mathbb{R}} f \overline{g}$ be the standard inner product in $L^2(\mathbb{R})$. The orthonormality relations

$$\langle \alpha_{k,n}, \alpha_{k',n'} \rangle = \delta_{kk'} \delta_{nn'} = \langle \beta_{k,n}, \beta_{k',n'} \rangle$$

follow from the orthonormality relations satisfied by the families (4.3), (4.4) and the fact that each product of the form $\Phi_{k,n} \Psi_{k',n'}$ is an odd function. The orthogonality relations

$$\langle \alpha_{k,n}, \beta_{k',n'} \rangle = 0$$
for all $k \in \mathbb{Z}$ and $n = 0, 1, 2, \ldots$, follow by the simple calculation:

$$
\langle \alpha_{k,n}, \beta_{k',n'} \rangle = \int_{\mathbb{R}} (\Phi_{k,n} \Phi_{k',n'} + \Phi_{k,n} \Psi_{k',n'}) + i(\Phi_{k,n} \Phi_{k',n'} - \Psi_{k,n} \Psi_{k',n'})
$$

$$
= 0 + \frac{i}{2}(\delta_{kk'}\delta_{nn'} - \delta_{kk}\delta_{nn'}) = 0.
$$

It remains to be shown that the above system is complete. In order to do this it clearly suffices to show that for each real valued $f \in L^2(\mathbb{R})$

$$
f = \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} \{ \langle f, \alpha_{k,n} \rangle \alpha_{k,n} + \langle f, \beta_{k,n} \rangle \beta_{k,n} \},
$$

(4.6)

where the convergence is in the $L^2(\mathbb{R})$ norm. But, writing $f = f^{(e)} + f^{(o)}$, with $f^{(e)}$ the even part and $f^{(o)}$ the odd part of $f$, we see that

$$
\sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} \langle f, \alpha_{k,n} \rangle \alpha_{k,n} =
$$

$$
\sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} \{ \langle f^{(e)}, \Psi_{k,n} \rangle - i\langle f^{(o)}, \Phi_{k,n} \rangle \} (\Psi_{k,n} + i\Phi_{k,n})
$$

from which we obtain

$$
\sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} \langle f, \alpha_{k,n} \rangle \alpha_{k,n} = \frac{1}{2} f + i \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} \{ \langle f^{(e)}, \Psi_{k,n} \rangle \Phi_{k,n} - \langle f^{(o)}, \Phi_{k,n} \rangle \Psi_{k,n} \}
$$

(4.7)

A completely analogous argument gives us

$$
\sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} \langle f, \beta_{k,n} \rangle \beta_{k,n} = \frac{1}{2} f + i \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} \{ \langle f^{(o)}, \Psi_{k,n} \rangle \Phi_{k,n} - \langle f^{(e)}, \Phi_{k,n} \rangle \Psi_{k,n} \}.
$$

(4.8)

Adding equalities (4.7) and (4.8) we obtain the desired result (4.6). $\blacksquare$

We now show that by modifying this basis slightly (multiplying its members by scalars of absolute value 1) we obtain an orthonormal basis of $L^2(\mathbb{R})$ that is generated by the single function $\gamma(\xi) \equiv i\alpha_{0,0}(\xi)$. More precisely, we have


Theorem (4.9). The functions
\[ \gamma_{k,n}(\xi) = 2^{k/2} e^{-i2^k n \xi} \gamma(2^k \xi), \]
for \( k, n \in \mathbb{Z} \), form an orthonormal basis of \( L^2(\mathbb{R}) \), where
\[ \gamma(\xi) = \frac{\text{sgn} \xi}{\sqrt{2\pi}} e^{i\xi b(|\xi|)}. \]

Proof: Let us put \( \alpha_{k,-n}(\xi) = \bar{\alpha}_{k,n-1}(\xi) = -i \beta_{k,n-1} \) for \( k \in \mathbb{Z} \) and \( n > 0 \). It then follows from Theorem (4.5) that \( \{\alpha_{k,n}\}, k, n \in \mathbb{Z} \), is an orthonormal basis for \( L^2(\mathbb{R}) \). Our theorem is established if we show
\[
\alpha_{0,n}(\xi) = (-1)^n (-i) e^{i n \xi} \gamma(\xi) \quad (4.10)
\]
for all \( n \in \mathbb{Z} \). To prove (4.10) assume, first, that \( n \geq 0 \), then
\[
\alpha_{0,n}(\xi) = \frac{1}{\sqrt{2\pi}} b(|\xi|) \cos \frac{2n+1}{2} (|\xi| - \pi) + i (\text{sgn} \xi) \sin \frac{2n+1}{2} (|\xi| - \pi)
\]
\[
= \frac{b(|\xi|)}{\sqrt{2\pi}} \begin{cases} e^{i(\xi-\pi)(2n+1)/2} & \text{if } \xi \geq 0 \\ e^{i(\xi+\pi)(2n+1)/2} & \text{if } \xi < 0 \end{cases}
\]
\[
= (-i)(-1)^n e^{i n \xi} \gamma(\xi).
\]
If \( n < 0 \), then \( \alpha_{0,n}(\xi) = \alpha_{0,-n-1}(\xi) = i \frac{\text{sgn} \xi}{\sqrt{2\pi}} (-1)^{-(n+1)} e^{i(n+1)\xi} e^{-i\xi} b(|\xi|) = (-i)(-1)^n e^{i n \xi} \gamma(\xi). \) Thus, (4.10) holds for all integers \( n \). ■

The promised wavelet basis is now immediately obtained from the functions in theorem (4.9) by applying the inverse Fourier transform to them. Let us carry out the details. We adopt the following definition of the Fourier transform of a function \( f \in L^1 \cap L^2 \):
\[ \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx. \]
The inverse Fourier transform formula is, in this case,
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi = \lim_{N \to \infty} \int_{-N}^{N} \hat{f}(\xi) e^{i\xi x} d\xi, \]
the limit being an \( L^2 \)-limit. A version of the Plancherel theorem, then tells us that if \( f \in L^2(\mathbb{R}) \), then
\[ \| \sqrt{2\pi} f \|_2 = \| \hat{f} \|_2, \quad (4.11) \]
and, if \( g \) is another function in \( L^2(\mathbb{R}) \), we have
\[ \langle f, g \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle. \quad (4.12) \]
Local Sine and Cosine Bases

Thus, if we define \( w \) by

\[
w(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \gamma(\xi) d\xi
\]

we have \( \hat{w} = \sqrt{2\pi} \gamma \),

\[
w_{k,n}(x) = 2^{-k/2} w(2^{-k} x - n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \gamma_{k,n}(\xi) d\xi,
\]

for \( k, n \in \mathbb{Z} \), and the collection \( \{w_{k,n}\} \) is an orthonormal basis of \( L^2(\mathbb{R}) \). Since \( \text{Supp} \hat{w} = \text{Supp} b(|\xi|) = [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \frac{8\pi}{3}] \) we see that \( w \) is, indeed, a mother function of the type described at the beginning of this section. That is, \( w \) generates a Lemarié-Meyer wavelet basis.

\section{Lemarié-Meyer wavelet bases with more general band limitations.}

In this section we shall examine to what extent we can generalize this construction if we consider other conditions on the support of \( \hat{w} \). We pose our problem in a way that is consistent with the basic properties of local sine/cosine bases. Instead of the interval \([\pi, 2\pi]\) let us consider the interval \([1, \lambda]\). Dilates by \( \lambda \), then, will then give us a covering of the right half-line:

\[
(0, \infty) = \bigcup_{k=-\infty}^{\infty} [\lambda^k, \lambda^{k+1}].
\]

We construct a bell function \( b \) associated with the interval \([1, \lambda]\), \( \epsilon = \frac{\lambda-1}{\lambda+1} \) and \( \epsilon' = \frac{\lambda(\lambda-1)}{(\lambda+1)(\lambda-1)} = \lambda \epsilon \). Observe that \( 1 + \epsilon = \lambda - \epsilon' \). The dilates \( b(\lambda^{-k} \xi) \) are then bell functions corresponding to the intervals \([\lambda^k, \lambda^{k+1}], k \in \mathbb{Z} \). Observe that adjacent intervals are compatible.

In complete analogy with the construction that led into Theorem (4.5), we let \( \alpha_{k,n}^\lambda = \Psi_{k,n}^\lambda + i \Phi_{k,n}^\lambda \) for \( k \in \mathbb{Z}, n \geq 0 \), and \( \alpha_{k,n}^\lambda = \overline{\alpha_{k,-n-1}^\lambda} \) if \( k \in \mathbb{Z} \) and \( n < 0 \), where

\[
\Psi_{k,n}^\lambda(\xi) = \sqrt{\frac{1}{2(\lambda - 1)}} \lambda^{\frac{k}{2}} b(\lambda^k |\xi|) \cos \pi(n + \frac{1}{2})(\frac{|\xi|\lambda^k - 1}{\lambda - 1}),
\]

\( k \in \mathbb{Z}, n \geq 0 \),

\[
\Phi_{k,n}^\lambda(\xi) = \sqrt{\frac{1}{2(\lambda - 1)}} \lambda^{\frac{k}{2}} b(\lambda^k |\xi|) (\text{sgn} \xi) \sin \pi(n + \frac{1}{2})(\frac{|\lambda^k|\xi - 1}{\lambda - 1}),
\]

\( k \in \mathbb{Z}, n \geq 0 \). The argument establishing Theorem (4.5) can be easily extended to show
Theorem (5.1). The collection of functions \( \{\alpha_{k,n}^\lambda\} \), \( k, n \in \mathbb{Z} \) is an orthonormal basis for \( L^2(\mathbb{R}) \).

Let us now examine if the functions \( \alpha_{k,n} = \alpha_{k,n}^\lambda \) are “generated”, via dilations (by integral powers of \( \lambda \)) and multiplications by \( e^{i\pi n \frac{\xi}{\lambda - 1}} \) \( (n \in \mathbb{Z}) \) applied to \( \alpha_{0,0} \) as was the case for the analogous basis in Theorem (4.9). We begin by trying to establish a formula similar to (4.10). If \( n \geq 0 \) we have

\[
\sqrt{2(\lambda - 1)} \alpha_{0,n}(\xi) = b(|\xi|) \begin{cases} 
    e^{i\pi(n + \frac{1}{2}) \frac{(\xi + 1)}{\lambda - 1}} , \xi > 0 \\
    e^{i\pi(n + \frac{1}{2}) \frac{(-\xi) + 1}{\lambda - 1}} , \xi < 0
\end{cases}
\]

where \( \rho = e^{-i\pi \frac{n}{\lambda - 1}} \). If \( n < 0 \) then

\[
\alpha_{0,n}(\xi) = \frac{1}{\sqrt{2(\lambda - 1)}} b(|\xi|) e^{-i\pi \frac{n}{\lambda - 1}} e^{i\pi \frac{\xi}{\lambda - 1}} e^{-i\pi(n + \frac{1}{2}) \frac{\text{sgn}(\xi)}{\lambda - 1}} e^{i\pi(n + \frac{1}{2}) \frac{\text{sgn}(\xi)}{\lambda - 1}}
\]

This shows

\[
\alpha_{0,n}(\xi) = \rho^{n \text{sgn}(\xi)} e^{i\pi n \frac{\xi}{\lambda - 1}} \alpha_{0,0}(\xi), \quad (5.2)
\]

for all \( n \in \mathbb{Z} \)

Thus, multiplication of \( \alpha_{0,0}(\xi) \) by \( e^{i\pi n \frac{\xi}{\lambda - 1}} \) gives us an orthonormal system, differing from \( \alpha_{0,n}(\xi) \) by a constant multiple of absolute value 1, provided \( \rho^{n \text{sgn}(\xi)} \) is a constant of absolute value 1 for all \( \xi \) (and each \( n \in \mathbb{Z} \)). But this means that \( \rho = e^{-i\pi \frac{n}{\lambda - 1}} \) must be real (either 1 or \(-1\)). Consequently, \( \sin \frac{\pi}{\lambda - 1} = 0 \) and it follows that \( \frac{\pi}{\lambda - 1} \) must be an integral multiple of \( \pi \). This and the condition \( \lambda > 1 \) force us to conclude that

\[
\lambda = 1 + \frac{1}{m} \quad (5.3)
\]

for \( m \) a positive integer. We thus obtain the following analog of Theorem (4.9):

Theorem (5.4). The functions

\[
\gamma_{k,n}^{(\lambda)}(\xi) = \gamma_{k,n}(\xi) = \lambda^k \rho^{-n \text{sgn}(\xi)} e^{-i\pi n \frac{\xi}{\lambda - 1}} \gamma(\lambda^k \xi), \quad (5.5)
\]

\( k, n \in \mathbb{Z} \), is an orthonormal basis of \( L^2(\mathbb{R}) \), where

\[
\gamma(\xi) = \frac{e^{i\pi \frac{\xi}{\lambda - 1}} b(|\xi|)}{\sqrt{2(\lambda - 1)}}
\]
and \( \lambda = 1 + \frac{1}{m}, m \) a positive integer.

If we let \( w(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \gamma(\xi) d\xi \), then \( w \) is a “mother function” that generates a wavelet basis (giving us a Multi Resolution Analysis)

\[
\{ w(x\lambda^{-k} - n\frac{\pi}{\lambda-1})\lambda^{-\frac{k}{2}} \},
\]

for all \( k, n \in \mathbb{Z} \), whenever \( \lambda = 1 + \frac{1}{m}, m \) a positive integer.

§6. Concluding remarks.

We repeat that the local bases we developed in §2, were introduced by Coifman and Meyer, and their use in obtaining the smooth wavelet bases were pointed out to us by these two authors. Some of the ideas involved in developing the properties of the local bases are also found in Malvar’s paper [4]. The particular emphasis on the rôle played by the projections \( P_t \), however, does not appear in Malvar’s paper [5] and is not prominent in the exposition [1] of Coifman and Meyer. As mentioned in the Abstract, one of our goals is to make this material accessible to the largest possible audience. One can develop a parallel treatment connected with the discrete sine and cosine transforms. We shall present this and its connection with the corresponding discrete version of the smooth wavelet bases in a subsequent paper.

References


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