THE SCATTERING TRANSFORM FOR THE BENJAMIN-ONO EQUATION

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Abstract. We use constructive methods to investigate the spectral theory of the Benjamin–Ono equation. Since the linearization series used previously is singular, we replace it with an improved series obtained by finite-rank renormalization. This introduces additional scattering data, which are shown to be dependent upon a single function, though not the usual one. We then prove the continuity of the direct and inverse scattering transforms defined by the improved series for small complex potentials. For all such potentials, the eigenvalues of the spectral problem cannot accumulate at 0. Rapidly decaying potentials have regular scattering data, prohibiting rapidly decaying solitons. In the selfadjoint case (real potentials), we obtain explicit cancellation of certain singularities. This leads to an alternate existence proof for the Cauchy problem for the equation. It also proves existence and gives estimates for some previously formal invariant quantities associated to the Benjamin–Ono hierarchy.

Introduction

This article investigates the direct and inverse scattering transforms for the Benjamin–Ono (BO) equation of hydrodynamics, which has attracted attention in recent years because of its remarkable algebraic properties. We will concern ourselves with the somewhat complicated analysis necessary for the rigorous interpretation of these properties. Our work extends results on the renormalization of singular integral equations, and provides constructive methods for the spectral theory of such equations.

Benjamin [Benj] and Ono [O] originally introduced the equation

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + H\left(\frac{\partial^2 u}{\partial x^2}\right), \qquad Hf(x) \stackrel{\text{def}}{=} p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt,$$

to describe the boundary between two immiscible fluids of differing densities. It was then discovered that this equation could be written in Lax form, a prerequisite for the application of the method of inverse scattering. However, the spectrum preserved under the resulting evolution proved unusually rich and complex, since it arose from a singular operator.

Several authors have approached this equation. Fokas and Ablowitz [FA] found the inverse spectral problem for BO. Beals and Coifman [BC1] observed its equivalence to a nonlocal $\bar{\partial}$ problem.

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They remarked that this feature appears in general in higher dimensional inverse scattering, and that the BO equation is therefore a good model to study. Matsuno [M] applied a variety of algebraic methods to obtain, among other things, the rational solutions of the BO equation, the Bäcklund transformations for the equation, and a hierarchy of nonlinear integro-differential equations related to the same Lax pair.

Anderson and Taflin obtain in a remarkable paper [AT] a formal power series linearization of the Benjamin–Ono equation. They observe that it can be interpreted as a distorted Fourier transform associated to a singular integral perturbation of $\frac{1}{i}\frac{d}{dx}$, from which they deduce a Lax pair for BO and obtain under suitable restrictions on the "scattering data" a power series for the inverse transform.

Unfortunately, both the direct and inverse problems considered by them are not analytic for generic potentials, leading to divergent series in general. See the remarks in the last section for a proof of this generic non-analyticity. Various formal conserved quantities obtained involve divergent integrals, and need to be analyzed with care. We were therefore led to study in great detail the spectral theory and the scattering or $\bar{\partial}$ transforms attached to singular integral perturbations of $\frac{1}{i}\frac{d}{dx}$. The methods for handling the difficulties encountered are inherently the same as those arising in the study of $\Delta + q$ in $\mathbf{R}^2$, and were used by Tsai in his thesis [T]. To obtain convergent series expansions, we are forced to renormalize the theory.

The main advantage of our renormalization will be to improve the behavior of the scattering data so that perturbation methods may be used to compute the inverse transform. This result is independent of self-adjointness. We can show that even for complex potentials, the eigenvalues of the spectral problem cannot accumulate near 0. Thus it is not the presence of bound states that creates the divergences at 0. Instead, we note that renormalization divides by an approximate determinant of the spectral problem, which can get arbitrarily small even though it never vanishes near 0.

When we improve the scattering transform, we introduce additional scattering data. These, however, are interdependent, and we show how they may be constructed from a knowledge of a single function. It is not the usual scattering function, and its regularity for rapidly decreasing potentials implies the nonexistence of rapidly decreasing BO solitons. The known rational soliton decays just slowly enough to evade this criterion.

The selfadjoint case, or assuming that q is real, allows additional identities and relations which may be used to cancel apparent singularities in the scattering data. Using these identities, we prove that some previously formal invariant integrals within the BO hierarchy indeed converge absolutely. The usual formal scattering data can be constructed as rational expressions in the improved data, and the singularities become removable because poles cancel. This parametrizes the formal data, allowing us to compute with them.

Finally, we will obtain a representation of the potential from the renormalized scattering data, and prove that this representation is a continuous map on a small neighborhood of the appropriate metric space of scattering data. We will then show that the various smallness assumptions fit together, and thereby solve the short-time Cauchy problem for the Benjamin–Ono equation. Such estimates prove continuity for the tangent maps we present relating the infinitesimal generators of evolutions in the BO hierarchy to the generators of the linear flows of scattering data. By including the selfadjointness assumption, we show the existence of solutions for all time from small real initial data.

1. Improved Eigenfunctions and Pre-Eigenfunctions

The linearization series developed for the Benjamin-Ono equation by Anderson and Taflin is obtained as the distorted Fourier transform for the operator

$$\frac{1}{i}\frac{d}{dx} - V(q)$$

where

$$V(q)f = P^+(qP^+(f)) - P^-(qP^-(f))$$

and

$$P^\pm(f) = \pm\frac{1}{2\pi}\int_0^{\pm\infty} e^{ix\xi}\hat{f}(\xi)\,d\xi.$$

To understand the spectral theory of this operator, as well as the fine behavior of this transform, we consider the pair of functions m^+, m^- satisfying the integral equations:

$$m^\pm(x,z) = 1 + \frac{1}{2\pi}\int_0^{\pm\infty} \frac{e^{ix\xi}\widehat{qm^\pm}(\xi,z)}{\xi - z}\,d\xi. \tag{1.1}$$

We will see later how to obtain eigenfunctions and express the distorted Fourier transform using $m^\pm$. It is convenient to write these equations in terms of Green's functions:

$$G_z^\pm(x) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{\pm\infty} \frac{e^{ix\xi}}{\xi - z}\,d\xi.$$

Denote by $\mathbf{G}_z$ the operator of convolution by G_z (in the x variable). Denote by $\mathbf{q}$ the operator of multiplication by the function $q = q(x)$. Then the integral equations in (1.1) may be written more succinctly as:

$$m^\pm(x,z) = 1 + \mathbf{G}_z^\pm \mathbf{q} m^\pm(x,z).$$

If $\|q\|_{L^2} < \epsilon$, Schwarz' inequality and the Banach fixed-point theorem show that there is some $\delta > 0$ such that the Picard series for $m^\pm(\cdot, z)$ will converge to a unique solution in $L^\infty(\mathbf{R})$ for all z with $|\Im z| > \delta$. As $\Im z \to 0$, however, any estimate on $\mathbf{G}_z^\pm$ will increase like $|\log z|$, and it becomes necessary to renormalize the series by subtracting away this singular behavior.

Let $\chi = \chi(\xi)$ be any smooth, even, compactly-supported real-valued function on $\mathbf{R}$ which is identically 1 in a neighborhood of $\xi = 0$. Let $l^\pm(z)$ be the pair of functions below, one for each sign:

$$l^\pm(z) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{\pm\infty} \frac{\chi(\xi)}{\xi - z}\,d\xi. \tag{1.2}$$

Then $l^\pm(z) \sim -\log|z|$ as $z \to 0$. Define a pair of improved Green's functions, one for each choice of sign:

$$G_z^{0\pm}(x) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{\pm\infty} \frac{e^{ix\xi} - \chi(\xi)}{\xi - z}\,d\xi = G_z^\pm(x) - l^\pm(z). \tag{1.3}$$

Depending on the choice of sign, $G_z^{0\pm}$ and $l^\pm(z)$ are holomorphic on the complement of the half-line $\{z \in \mathbf{R} : \pm z \geq 0\}$.

Observe that for any $\delta > 0$,

$$\lim_{\substack{z\to\infty \\ |\Im z|>\delta}} G_z^{0\pm}(x) = 0, \qquad \text{and} \qquad \lim_{\substack{z\to\infty \\ |\Im z|>\delta}} l^\pm(z) = 0.$$

Thus the maximum principle applies to $G_z^{0\pm}$ and $l^\pm(z)$, so they are dominated by the supremum of their boundary values at the appropriate half-line. To avoid cumbersome notation, suppress the superscript$^\pm$ and introduce the convention

$$G_z^0(x) = \begin{cases} G_z^{0+}(x), & \text{if } \Re z > 0, \\ G_z^{0-}(x), & \text{if } \Re z < 0, \end{cases}$$

$$l(z) = \begin{cases} l^+(z), & \text{if } \Re z > 0, \\ l^-(z), & \text{if } \Re z < 0. \end{cases}$$

Then boundary values are defined at $\zeta \in \mathbf{R}$ by:

$$G^0_{\zeta\pm} \stackrel{\text{def}}{=} \lim_{\epsilon\to 0^+} G^0_{\zeta\pm i\epsilon}(x), \qquad \text{and} \qquad l(\zeta\pm) \stackrel{\text{def}}{=} \lim_{\epsilon\to 0^+} l(\zeta \pm i\epsilon).$$

The operator boundary values $\mathbf{G}^0_{\zeta\pm}$ exist as strong limits in the weighted L^p topology to be described below. In fact there is a pointwise estimate for the boundary values of G^0_z:

Lemma 1-1. *Let G^0_z be defined as above. Then there is some constant C independent of ζ and x such that both limiting values of G^0_z satisfy the following estimate:*

$$|G^0_{\zeta\pm}(x)| \leq C \log(|x| + \frac{1}{|x|})$$

Proof. The original Green function $G_z(x)$ satisfies the classical estimates below, obtained by evaluating a contour integral in three pieces:

$$|G_z(x)| = |G_{xz}(1)| \leq C\left| \int_0^{|xz|/2} \frac{e^{it}}{t - xz}\,dt + \int_\Gamma \frac{e^{it}}{t-xz}\,dt + \int_{3|xz|/2}^\infty \frac{e^{it}}{t-xz}\,dt \right|,$$

where Γ is a small semicircular arc that dodges the singularity at $t = |xz|$. The first two terms are uniformly bounded in $|xz|$ since the length of the curve is $c|xz|$ and the integrand is dominated by $c/|xz|$. The third term is bounded for $|xz| > \frac{1}{2}$, by integration by parts. Otherwise, consider its decomposition into $\int_{3|xz|/2}^1 + \int_1^\infty$. The integral to infinity is bounded, and we are left with

$$\int_{3|xz|/2}^1 \frac{e^{it}}{t-xz}\,dt = \int_{3|xz|/2}^1 \left(\frac{e^{it}}{t-xz} - \frac{1}{t}\right) dt + \log(\frac{3}{2}|xz|).$$

The integral is bounded, leading to

$$G_z(x) = \begin{cases} B(xz), & \text{if } |xz| > \frac{1}{2}, \\ B(xz) + \log xz, & \text{if } |xz| \leq \frac{1}{2}, \end{cases}$$

with B bounded. For $|z| < 1$, $\log|xz| - l(z) - \log|x|$ is bounded. For $|z| \geq 1$, if $|xz| \leq 1/2$, then $\log|xz| \leq \log|x|$). □

The significance of this lemma is that $G^{0\pm}_z \in L^p_{\text{loc}}(dx)$ for every $1 \leq p < \infty$, with norm uniformly bounded over $z \in \mathbf{C} \setminus \mathbf{R}^\pm$. Contrast this with $G^\pm_z$, which belongs to $L^p_{\text{loc}}(dx)$ for $1 \leq p < \infty$ but with a norm that blows up as $z \to 0$.

Convolution with G^0_z is therefore bounded uniformly in z on certain weighted L^p spaces. Let $w(x) = 1 + |x|$. It is elementary to show that for any $n > 0$, $w^n(x-y) \leq w^n(x)w^n(y)$, so w^n is a weight function in the classical sense.

Lemma 1-2. *Suppose $1 \leq p \leq \infty$. If $w^n f \in L^p(\mathbf{R})$ for some $n > (p-1)/p$, then there is a constant C depending only upon n for which $|\mathbf{G}^0_z f(x)| \leq C w^n(x) \|w^n f\|_{L^p}$.*

Proof. Multiplying and dividing by the weight w^n yields the estimate:

$$\begin{aligned} |\mathbf{G}^0_z f(x)| &\leq \int_{\mathbf{R}} |G^0_z(x-y) f(y)|\,dy \\ &= w^n(x) \int_{\mathbf{R}} \frac{|G^0_z(x-y)|}{w^n(x)w^n(y)} |w^n(y) f(y)|\,dy \\ &\leq w^n(x) \int_{\mathbf{R}} \frac{|G^0_z(x-y)|}{w^n(x-y)} |w^n(y) f(y)|\,dy. \end{aligned}$$

By the previous lemma, if $n > (p-1)/p \stackrel{\text{def}}{=} 1/p'$, then $w^{-n}G_z^0$ belongs to $L^{p'}(\mathbf{R})$. By assumption, $w^n f$ belongs to $L^p(\mathbf{R})$, so the convolution is bounded and we have

$$|\mathbf{G}_z^0 f(x)| \leq w^n(x) \left\|w^{-n}G_z^0\right\|_{L^{p'}} \|w^n f\|_{L^p}.$$

Taking $C = \sup_z \left\|w^{-n}G_z^0\right\|_{L^{p'}}$, which depends only upon n, completes the proof. □

Corollary 1-3. *The operators $\mathbf{G}_{\zeta\pm}^0$ are strong limits in $B(w^{-n}L^p, w^n L^\infty)$ of operators $\mathbf{G}_{\zeta\pm i\epsilon}^0$, as $\epsilon \to 0^+$, for every $n > (p-1)/p$.*

Proof. Looking ahead to the proof of Proposition 1-7, we see that $G_{\zeta\pm i\epsilon}^0(x)$ is Hölder continuous in ϵ for each fixed x, so that $\mathbf{G}_{\zeta\pm i\epsilon}^0 f(x) \to \mathbf{G}_{\zeta\pm}^0 f(x)$ as $\epsilon \to 0$ by the Lebesgue dominated convergence theorem. This convergence is uniform in x on compact sets.

Now, if $w^n f \in L^p$, then also $w^{n-\delta} f \in L^p$, where $\delta > 0$ can be so small that $n - \delta > (p-1)/p$. But then $w^{-n+\delta}(x)\mathbf{G}_{\zeta\pm i\epsilon}^0 f(x)$ is bounded uniformly in x, ζ and ϵ. This extra decay ensures that $w^{-n}(x)\mathbf{G}_{\zeta\pm i\epsilon}^0 f(x) \to 0$ as $|x| \to \infty$ uniformly for all ϵ. Thus $w^{-n}\mathbf{G}_{\zeta\pm i\epsilon}^0 f \to w^{-n}\mathbf{G}_{\zeta\pm}^0 f$ in L^∞. □

Corollary 1-4. *Suppose $w^n q \in L^p$ for some $n > (p-1)/p$. Then the family of operators $z \mapsto \mathbf{G}_z^{0\pm}\mathbf{q}$ is bounded analytic in $B(L^\infty, w^n L^\infty)$ for $z \in \mathbf{C} \setminus \mathbf{R}^\pm$.*

Proof. Analyticity is clear, since $\mathbf{q} : L^\infty \to w^{-n}L^p$ and does not depend upon z. Boundedness follows from the maximum principle, since the operator norm of $\mathbf{G}_z^{0\pm}\mathbf{q}$ is dominated by the supremum of its boundary values, which are estimated in Lemma 1-2 and Corollary 1-3. □

Thus, given a small potential q with sufficient decay , it is possible to iterate the map $\mathbf{G}_z^0\mathbf{q}$ to construct slowly increasing functions analytic in z off the real axis. These satisfy an estimate in x uniformly for all z. This is the basic existence result for the improved scattering equation.

Proposition 1-5. *Select $1 \leq p \leq \infty$, and set $n > (p-1)/p$. If $w^{2n}q$ has small norm in $L^p(\mathbf{R})$, then for every $z \in \mathbf{C} \setminus \mathbf{R}^\pm$, there exist unique solutions $m^{0\pm} \in w^n L^\infty$ to the integral equations*

$$m^{0\pm}(x,z) = 1 + \mathbf{G}_z^{0\pm}\mathbf{q}m^{0\pm}(x,z). \tag{1.4}$$

In addition, if $w^{n+1}q$ has small norm in $L^p(\mathbf{R})$, then for every $z \in \mathbf{C} \setminus \mathbf{R}^\pm$, there exist unique solutions $p^{0\pm} \in w^n L^\infty$ to the integral equations

$$p^{0\pm}(x,z) = ix + \mathbf{G}_z^{0\pm}\mathbf{q}p^{0\pm}(x,z). \tag{1.5}$$

Furthermore, the maps $q \mapsto m^{0\pm}$ and $q \mapsto p^{0\pm}$ are uniformly bounded in z for each q in some neighborhood of 0. Finally, the maps $q \mapsto qm^{0\pm}$ and $q \mapsto qp^{0\pm}$ are Lipschitz continuous in q at $q = 0$, with Lipschitz constant bounded uniformly in z.

Proof. For simplicity, suppress the superscripts$^\pm$. Consider first the integral equation for gm^0, where $g = g(x)$ is some measurable function. Observe that whenever the series converges,

$$gm^0 = g + g\mathbf{G}_z^0\mathbf{q}m^0 = g + g\mathbf{G}_z^0 q + g\mathbf{G}_z^0\mathbf{q}\mathbf{G}_z^0 q + \ldots \tag{1.6}$$

By Corollary 1-4, there is a constant C_n independent of both x and z for which $|\mathbf{G}_z^0 q(x)| \leq C_n w^n(x)\|w^n q\|_{L^p}$. But then $\|w^n q\mathbf{G}_z^0 q\|_{L^p} < C_n\|w^{2n}q\|_{L^p}\|w^n q\|_{L^p} < \infty$. This allows another application of $\mathbf{G}_z^0$, so it is possible to iterate $\mathbf{G}_z^0\mathbf{q}$. Thus for arbitrary g,

$$\left|g\left(\mathbf{G}_z^0\mathbf{q}\right)^k 1\right|(x,z) < C_n^k|w^n(x)g(x)| \left\|w^{2n}q\right\|_{L^p}^{k-1} \|w^n q\|_{L^p}. \tag{1.7}$$

Hence, for small enough q the series in Eq.(1.6) converges pointwise geometrically, uniformly at every $z \in \mathbf{C} \setminus \mathbf{R}^{\pm}$. Moreover, noting that $\|w^n q\|_{L^p} \leq \|w^{2n} q\|_{L^p}$, it yields the estimate

$$|g(x) m^0(x,z)| < \frac{|w^n(x) g(x)|}{1 - C_n \|w^{2n} q\|_{L^p}}.$$

Taking $g = w^{-n}$ shows that for small q a unique solution m^0 exists and grows slowly in the sense that $|m^0(x,z)| < M(1+|x|)^n$ for some constant M independent of z.

Taking $g = q$ and arbitrary p shows that

$$\|q m^0\|_{L^p} < \frac{\|w^n q\|_{L^p}}{1 - C_n \|w^{2n} q\|_{L^p}},$$

proving Lipschitz continuity at $q = 0$.

For p^0, the iteration of $\mathbf{G}_z^0 \mathbf{q}$ yields

$$gp^0 = ix\, g + g\mathbf{G}_z^0 \mathbf{q} p^0 = ix\, g + g\mathbf{G}_z^0 (ix\, q) + g\mathbf{G}_z^0 \mathbf{q} \mathbf{G}_z^0 (ix\, q) + \dots \tag{1.8}$$

The L^p norm of $ix\, w^n(x) q(x)$ is majorized by $\|w^{n+1} q\|_{L^p}$. Repeating the previous argument yields the estimate

$$\left| g \left(\mathbf{G}_z^0 \mathbf{q} \right)^k (ix) \right| (x,z) < C_n^k |w^n(x) g(x)| \left\| w^{2n} q \right\|_{L^p}^{k-1} \left\| w^{n+1} q \right\|_{L^p}. \tag{1.9}$$

Hence, for small enough q the series in Eq.(1.8) converges geometrically, again uniformly for all $z \in \mathbf{C} \setminus \mathbf{R}^{\pm}$. Moreover, it satisfies an estimate similar to the one for m^0:

$$|g(x) p^0(x,z)| < |ix\, g(x)| + \frac{C_n \, \|w^{n+1} q\|_{L^p} \, |w^n(x) g(x)|}{1 - C_n \|w^{2n} q\|_{L^p}}.$$

Taking $g = w^{-n-1}$ shows that for small q a unique solution p^0 exists and grows slowly in the same sense as m^0: there is a constant N independent of z such that $|p^0(x,z)| < N(1+|x|)^{n+1}$.

Observe that both $|ix\, q|$ and $|w^n q|$ are dominated by $|w^{n+1} q|$. Thus, taking $g = q$ shows that

$$\|q p^0\|_{L^p} < \frac{\|w^{n+1} q\|_{L^p}}{1 - C_n \|w^{2n} q\|_{L^p}}, \tag{1.10}$$

proving Lipschitz continuity at $q = 0$. □

There is another family of functions important to the analysis of this problem, whose properties derive from a similar construction. These arise from the jump discontinuities in z of the functions $m^{0\pm}(x,z)$ across $\mathbf{R}^{\pm}$, and therefore are only defined at real values of z, which shall be renamed ζ to emphasize this distiction.

For real ζ, denote by $\mathbf{e}$ and $\mathbf{e}^*$ the operations of multiplication by $e^{ix\zeta}$ and $e^{-ix\zeta}$, respectively. Then it is easy to see that for any $z \in \mathbf{C} \setminus \mathbf{R}^{\pm}$, the operators $\mathbf{e}^* \mathbf{G}_z^{0\pm} \mathbf{e}\mathbf{q}$ satisfy the same estimates as $\mathbf{G}_z^{0\pm} \mathbf{q}$. Thus for q small in the sense of Proposition 1-5, there are unique solutions $N^{\pm}(x,z,\zeta)$ to the integral equations

$$N^{\pm} = 1 + \mathbf{e}^* \mathbf{G}_z^{0\pm} \mathbf{e}\mathbf{q} N^{\pm}. \tag{1.11}$$

As before, $|w^{-n} N^{\pm}|$ will be uniformly bounded in x, z, and ζ.

Now define two functions

$$n^{0\pm}(x,\zeta) \stackrel{\text{def}}{=} \lim_{\epsilon \to 0^+} N^{\pm}(x, \zeta - i\epsilon, \zeta). \tag{1.12}$$

Because of the properties of $N^{\pm}$, the following is immediate:

Proposition 1-6. *Select $1 \le p \le \infty$, and set $n > (p-1)/p$. If $w^{2n}q$ has small norm in $L^p(\mathbf{R})$, then for every $\zeta \in \mathbf{R}$ the function $n^{0\pm}(x,\zeta)$ exists in $w^n L^\infty$. Furthermore, the maps $q \mapsto n^{0\pm}$ are uniformly bounded in ζ for each q in some neighborhood of 0. Finally, the maps $q \mapsto qn^{0\pm}$ are Lipschitz continuous in q at $q=0$, with Lipschitz constant bounded uniformly in ζ.* □

These $n^{0\pm}$ are related to the eigenfunctions of the selfadjoint operator $-id/dx - V_q$ introduced by Anderson and Taflin in [AT]. Improving the operator V_q as above allows these *improved eigenfunctions* to be constructed with good estimates. Indeed, they have a smooth dependence upon the spectral variable, as will be shown below. The functions $p^{0\pm}$ and $m^{0\pm}$ are holomorphic in the upper and lower half z-plane with a jump across the real axis which is (almost) an improved eigenfunction, so they shall be called *pre-eigenfunctions*.

These improved spectral objects inherit the same smoothness in the z-variable that G_z gains when it is modified into G_z^0:

Proposition 1-7. *If q is small with sufficient decay (as in Proposition 1-5), then for any $\delta < 1$ the functions $m^{0\pm}(x,z)$, $n^{0\pm}(x,z)$ and $p^{0\pm}(x,z)$ are Hölder continuous of degree δ in the z-variable.*

Proof. Since G_z^0 is the Cauchy integral of a bounded function, we can use an estimate for the Hilbert transform of $[e^{ix\xi} - \chi(\xi)]\mathbf{1}_{\mathbf{R}\pm}(\xi)$. For every $x \in \mathbf{R}$, this is a Lipschitz continuous function of ξ. By a well-known result, therefore, for every $\delta < 1$ there is a constant C_δ, independent of ζ and x, such that

$$|G^0_{\zeta\pm}(x) - G^0_{\zeta'\pm}(x)| \le C_\delta \big\| [e^{ix\cdot} - \chi]\mathbf{1}_{\mathbf{R}\pm} \big\|_\delta |\zeta - \zeta'|^\delta \le C_\delta (1 + |x|^\delta)|\zeta - \zeta'|^\delta,$$

where $\|\cdot\|_\delta$ is the Hölder norm. The decay of q at infinity will kill off the growth of this estimate in x.

The Hölder continuity of $m^{0\pm}$ follows from that of $G_z^{0\pm}$:

$$m^{0\pm}(x,z) - m^{0\pm}(x,z') = \left(\mathbf{G}_z^{0\pm}\mathbf{q} - \mathbf{G}_{z'}^{0\pm}\mathbf{q}\right) m^{0\pm}(x,z') + \mathbf{G}_z^{0\pm}\mathbf{q}\left(m^{0\pm}(x,z) - m^{0\pm}(x,z')\right).$$

The first term is $O(|z-z'|^\delta)$ for all $0 < \delta < 1$. Since $\mathbf{G}_z^{0\pm}\mathbf{q}$ is a contraction, $m^{0\pm}(x,z) - m^{0\pm}(x,z')$ is likewise $O(|z-z'|^\delta)$. Similar arguments work for $p^{0\pm}$ and $n^{0\pm}$, the latter having z restricted to the real axis. □

In passing, note that the iteration $(I - \mathbf{G}_z^{0\pm}\mathbf{q})^{-1}(x^k)$ converges for $k > 1$, so long as $w^{n+k}q$ belongs to $L^p(\mathbf{R})$. This means that q must have sufficient decay to get things started. For $|q(x)| \sim w(x)^{-j}$ as $x \to \infty$, the requirement is $|w^{n+k-j}|^p \in L^1(\mathbf{R})$, or that $pn + p(k-j) < -1$. Since $pn > p-1$, this implies $k < j-1$. A consequence of this estimate is that the rational soliton solution $q = 2(1+(x-t)^2)^{-1}$ has too little decay as $x \to \infty$ for the eigenfunctions to be regular. There are, conversely, no small rapidly decaying solitons for the Benjamin–Ono equation, for they would correspond to singular eigenfunctions.

2. Scattering Data and Departure from Holomorphy

The pre-eigenfunctions and eigenfunctions satisfy nonlinear differential equations in the spectral variable. These may be written as simple linear equations with coefficients, or *scattering data*, that are functionals in the potential. It will be shown that the solutions to these equations are determined uniquely by the coefficients.

The *(improved) scattering data* are integral transforms of the potential against the improved eigenfunctions:

$$\alpha^{0\pm}(z) \stackrel{\text{def}}{=} \int_{\mathbf{R}} q(x) m^{0\pm}(x,z)\,dx, \tag{2.1}$$

$$s^{0\pm}(\zeta) \stackrel{\text{def}}{=} \int_{\mathbf{R}} e^{-ix\zeta} q(x) n^{0\mp}(x,-\zeta)\,dx, \tag{2.2}$$

$$\beta^{0\pm}(z) \stackrel{\text{def}}{=} \int_{\mathbf{R}} q(x) p^{0\pm}(x,z)\,dx, \tag{2.3}$$

$$r^{0\pm}(\zeta) \stackrel{\text{def}}{=} \int_{\mathbf{R}} ix\, e^{-ix\zeta} q(x) n^{0\mp}(x,-\zeta)\,dx. \tag{2.4}$$

Using the existence results of the last section makes the following immediate:

Proposition 2-1. *For any $n > 0$, the maps $q \mapsto \alpha^{0\pm}$ and $q \mapsto s^{0\pm}$ exist and are continuous (or even analytic) between small neighborhoods of 0 in $w^{-n}L^1(\mathbf{R})$ and L^∞. Likewise, the maps $q \mapsto \beta^{0\pm}$ and $q \mapsto r^{0\pm}$ exist and are continuous between small neighborhoods of 0 in $w^{-n-1}L^1(\mathbf{R})$ and L^∞.*

Proof. Choose $p = 1$ in Proposition 1-5 and Proposition 1-6. □

Because of the symmetries of the operator $\mathbf{G}_z^{0\pm}$, these functions have other integral representations obtained by transposition. In particular, there are the following relations among transposes:

$$G_z^{\pm}(-x) = G_{-z}^{\mp}(x) \implies {}^t\mathbf{G}_z^{\pm} = \mathbf{G}_{-z}^{\mp}, \tag{2.5}$$

since χ is an even function, $l^{\pm}(z) = l^{\mp}(-z)$. Therefore,

$$G_z^{0\pm}(-x) = G_{-z}^{0\mp}(x) \implies {}^t\mathbf{G}_z^{0\pm} = \mathbf{G}_{-z}^{0\mp} \tag{2.6}$$

With enough decay in q, the integrals defining the scattering data above are absolutely convergent and allow transposition as follows:

$$\begin{aligned}
\alpha^{0\pm}(z) &= \int_{\mathbf{R}} q(x) m^{0\pm}(x,z)\,dx = \int_{\mathbf{R}} q(x) \left(I - \mathbf{G}_z^{0\pm}\mathbf{q}\right)^{-1} 1\,dx \\
&= \int_{\mathbf{R}} \left(I - {}^t(\mathbf{G}_z^{0\pm}\mathbf{q})\right)^{-1} q(x)\,dx \\
&= \int_{\mathbf{R}} \left(I - \mathbf{q}\mathbf{G}_{-z}^{0\mp}\right)^{-1} q(x)\,dx \\
&= \int_{\mathbf{R}} q(x) \left(I - \mathbf{G}_{-z}^{0\mp}\mathbf{q}\right)^{-1} 1\,dx \\
&= \int_{\mathbf{R}} q(x) m^{0\mp}(x,-z)\,dx \\
&= \alpha^{0\mp}(-z).
\end{aligned} \tag{2.7}$$

For $s^{0\pm}$, it is necessary to keep track of the direction of the limit:

$$
\begin{aligned}
s^{0\pm}(\zeta) &= \int_{\mathbf{R}} e^{-ix\zeta} q(x) n^{0\mp}(x,-\zeta)\,dx \\
&= \lim_{\epsilon\to 0^+} \int_{\mathbf{R}} e^{-ix\zeta} q(x) \left(I - \mathbf{e}\mathbf{G}^{0\mp}_{(-\zeta)-i\epsilon}\mathbf{e}^*\mathbf{q}\right)^{-1} 1\,dx \\
&= \lim_{\epsilon\to 0^+} \int_{\mathbf{R}} \left(I - {}^t(\mathbf{e}\mathbf{G}^{0\mp}_{(-\zeta)-i\epsilon}\mathbf{e}^*\mathbf{q})\right)^{-1} \mathbf{e}^* q(x)\,dx \\
&= \lim_{\epsilon\to 0^+} \int_{\mathbf{R}} \left(I - \mathbf{q}\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta+i\epsilon}\mathbf{e}\right)^{-1} \mathbf{e}^* q(x)\,dx \\
&= \lim_{\epsilon\to 0^+} \int_{\mathbf{R}} e^{-ix\zeta} q(x) \left(I - \mathbf{G}^{0\pm}_{\zeta+i\epsilon}\mathbf{q}\right)^{-1} 1\,dx \\
&= \lim_{\epsilon\to 0^+} \int_{\mathbf{R}} e^{-ix\zeta} q(x) m^{0\pm}(x,\zeta+i\epsilon)\,dx \\
&= \int_{\mathbf{R}} e^{-ix\zeta} q(x) m^{0\pm}(x,\zeta+)\,dx.
\end{aligned} \tag{2.8}
$$

Similar formulas hold for the other two functions:

$$
\beta^{0\pm}(z) = \int_{\mathbf{R}} ix\, q(x) m^{0\mp}(x,-z)\,dx, \tag{2.9}
$$

$$
r^{0\pm}(\zeta) = \int_{\mathbf{R}} e^{-ix\zeta} q(x) p^{0\pm}(x,\zeta+)\,dx. \tag{2.10}
$$

The scattering data used in the formal schemes is the above with no renormalization, namely with $\chi = 0$. This gives functions $\alpha^\pm$ and $\beta^\pm$ defined on $\mathbf{C}\backslash\mathbf{R}^\pm$, $s^\pm$ and $r^\pm$ defined on $\mathbf{R}^\pm$, $m^\pm$ and $p^\pm$ defined on $\mathbf{R}\times(\mathbf{C}\backslash\mathbf{R}^\pm)$, and $n^\pm$ defined on $\mathbf{R}\times\mathbf{R}^\pm$, whenever they exist. These may be related to the improved functions above.They are rational expressions in the improved eigenfunctions and improved scattering data. To show this, it is necessary to assume that the original integral equation have solutions, namely that $I - \mathbf{G}^\pm_z\mathbf{q}$ and $I - \mathbf{e}^*\mathbf{G}^\pm_z\mathbf{e}\mathbf{q}$ are invertible. Then,

$$
\begin{aligned}
m^{0\pm}(x,z) &= 1 + \mathbf{G}^{0\pm}_z \mathbf{q} m^{0\pm}(x,z) \\
&= 1 + \mathbf{G}^{\pm}_z \mathbf{q} m^{0\pm}(x,z) - l^\pm(z) \int_{\mathbf{R}} q(y) m^{0\pm}(y,z)\,dy \\
&= 1 + \mathbf{G}^{\pm}_z \mathbf{q} m^{0\pm}(x,z) - l^\pm(z)\alpha^{0\pm}(z), \\
\implies m^{0\pm}(x,z) &= (I - \mathbf{G}^\pm_z\mathbf{q})^{-1}\left[1 - l^\pm(z)\alpha^{0\pm}(z)\right] \\
&= \left[1 - l^\pm(z)\alpha^{0\pm}(z)\right](I - \mathbf{G}^\pm_z\mathbf{q})^{-1} 1 = \left[1 - l^\pm(z)\alpha^{0\pm}(z)\right] m^\pm(x,z),
\end{aligned}
$$

$$
\implies m^\pm(x,z) = \frac{m^{0\pm}(x,z)}{1 - l^\pm(z)\alpha^{0\pm}(z)}. \tag{2.11}
$$

Similar relations hold between the scattering data and the improved scattering data:

$$
\begin{aligned}
\alpha^\pm(z) &= \int_{\mathbf{R}} q(x) m^\pm(x,z)\,dx = \int_{\mathbf{R}} q(x) \frac{m^{0\pm}(x,z)}{1 - l^\pm(z)\alpha^{0\pm}(z)}\,dx \\
&= \frac{\alpha^{0\pm}(z)}{1 - l^\pm(z)\alpha^{0\pm}(z)},
\end{aligned} \tag{2.12}
$$

$$s^{\pm}(\zeta) = \frac{s^{0\pm}(\zeta)}{1 - l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)}, \tag{2.13}$$

$$\beta^{\pm}(z) = \frac{\beta^{0\pm}(z)}{1 - l^{\pm}(z)\alpha^{0\pm}(z)}. \tag{2.14}$$

The relations among the remaining functions are slightly more complicated, but quite similar:

$$\begin{aligned}
p^{0\pm}(x,z) &= ix + \mathbf{G}_z^{0\pm}\mathbf{q}p^{0\pm}(x,z) \\
&= ix + \mathbf{G}_z^{\pm}\mathbf{q}p^{0\pm}(x,z) - l^{\pm}(z)\int_{\mathbf{R}} q(y)p^{0\pm}(y,z)\,dy \\
&= ix + \mathbf{G}_z^{\pm}\mathbf{q}p^{0\pm}(x,z) - l^{\pm}(z)\beta^{0\pm}(z), \\
\implies p^{0\pm}(x,z) &= (I - \mathbf{G}_z^{\pm}\mathbf{q})^{-1}\left(ix - l^{\pm}(z)\beta^{0\pm}(z)\right) \\
&= p^{\pm}(x,z) - l^{\pm}(z)\beta^{0\pm}(z)m^{\pm}(x,z),
\end{aligned}$$

$$\implies p^{\pm}(x,z) = p^{0\pm}(x,z) + \frac{l^{\pm}(z)\beta^{0\pm}(z)m^{0\pm}(x,z)}{1 - l^{\pm}(z)\alpha^{0\pm}(z)}, \tag{2.15}$$

$$\begin{aligned}
n^{0\pm}(x,\zeta) &= 1 + \mathbf{e}^*\mathbf{G}_{\zeta_-}^{0\pm}\mathbf{qe}n^{0\pm}(x,\zeta) \\
&= 1 + \mathbf{e}^*\mathbf{G}_{\zeta_-}^{\pm}\mathbf{qe}n^{0\pm}(x,\zeta) - e^{-ix\zeta}l^{\pm}(\zeta-)\int_{\mathbf{R}} e^{iy\zeta}q(y)n^{0\pm}(y,\zeta)\,dy \\
&= 1 + \mathbf{e}^*\mathbf{G}_{\zeta_-}^{\pm}\mathbf{qe}n^{0\pm}(x,\zeta) - e^{-ix\zeta}l^{\pm}(\zeta-)s^{0\mp}(-\zeta), \\
\implies n^{0\pm}(x,\zeta) &= (I - \mathbf{e}^*\mathbf{G}_{\zeta_-}^{\pm}\mathbf{qe})^{-1}\left(1 - e^{-ix\zeta}l^{\pm}(\zeta-)s^{0\mp}(-\zeta)\right) \\
&= n^{\pm}(x,\zeta) - e^{-ix\zeta}l^{\pm}(\zeta-)s^{0\mp}(-\zeta)m^{\pm}(x,\zeta-),
\end{aligned}$$

$$\implies n^{\pm}(x,\zeta) = n^{0\pm}(x,\zeta) + \frac{e^{-ix\zeta}l^{\pm}(\zeta-)s^{0\mp}(-\zeta)m^{0\pm}(x,\zeta-)}{1 - l^{\pm}(\zeta-)\alpha^{0\pm}(\zeta-)}, \tag{2.16}$$

$$\begin{aligned}
r^{\pm}(\zeta) &= \int_{\mathbf{R}} e^{-ix\zeta}q(x)p^{\pm}(x,\zeta+)\,dx \\
&= \int_{\mathbf{R}} e^{-ix\zeta}q(x)\left[p^{0\pm}(x,\zeta+) + \frac{l^{\pm}(\zeta+)\beta^{0\pm}(\zeta+)m^{0\pm}(x,\zeta+)}{1 - l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)}\right]dx \\
&= r^{0\pm}(\zeta) + \frac{l^{\pm}(\zeta+)\beta^{0\pm}(\zeta+)s^{0\pm}(\zeta)}{1 - l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)}.
\end{aligned} \tag{2.17}$$

In all of these equations, the only singularity that appears is the possible vanishing of $1 - l^{\pm}\alpha^{0\pm}$ in the various denominators. For small q, this quantity is an approximate determinant of the operator $I - \mathbf{G}_z^{\pm}\mathbf{q}$. The renormalization to an improved series amounts to dividing the operator by its determinant, although this only makes sense in limited generality. For q of sufficiently rapid decay, the operator is a Hilbert-Schmidt perturbation of I, depending holomorphically on $z \in \mathbf{C} \setminus \mathbf{R}$. Thus by the analytic Fredholm theorem, $1 - l^{\pm}\alpha^{0\pm}$ vanishes like a power of $z - z_0$ at a discrete set of roots z_0.

In the case of real-valued q, there are identities which will be used in a later chapter to show that the singularity introduced by the denominator is removable. That gives a direct proof that the scattering transform is continuous in the selfadjoint case. This fact was proved by Anderson and Taflin by other methods.

Analogous to the Fourier transform, small smooth potentials give rapidly decreasing scattering data.

Proposition 2-2. *Fix $n > 0$, and let $K > 0$ be an integer. If $w^n d^k q/dx^k$ is small in $L^1(\mathbf{R})$ for all $0 \le k \le K$, then $w^K(\zeta)s^{0\pm}(\zeta)$ is bounded for all $\zeta \in \mathbf{R}$. In addition, if $w^{n+1}d^k q/dx^k$ is small in $L^1(\mathbf{R})$ for all $0 \le k \le K$, then $w^K(\zeta)r^{0\pm}(\zeta)$ is bounded for all $\zeta \in \mathbf{R}$.*

Proof. This is proved by induction on K. Proposition 1-5 and Proposition 1-6 establish the result for $K = 0$. Consider the following expression, which follows from Eq.(2.8):

$$|\zeta^K s^{0\pm}(\zeta)| = \left| \int_{\mathbf{R}} \left(\frac{d^K e^{-ix\zeta}}{dx^K} \right) q(x) m^{0\pm}(x, \zeta+)\, d\zeta \right|.$$

If q and its derivatives have the stated decay, the derivatives may be transposed onto $q(x)m^{0\pm}(x, \zeta+)$. The resulting integral will be bounded if $m^{0\pm}(x, \zeta+)$ is smooth. But for $K > 0$ its derivatives satisfy the integral equation

$$\begin{aligned} \frac{d^K}{dx^K} m^{0\pm}(x, \zeta+) &= \mathbf{G}^0_{z+} \frac{d^K(\mathbf{q}m^{0\pm})}{dx^K}(x, \zeta+) \\ &= \mathbf{G}^0_{z+} \sum_{k=0}^{K-1} \binom{K}{k} \mathbf{q}^{(K-k)} \frac{d^k m^{0\pm}}{dx^k}(x, \zeta+) + \mathbf{G}^0_{z+}\mathbf{q} \frac{d^K m^{0\pm}}{dx^K}(x, \zeta+). \end{aligned}$$

The sum is bounded by inductive hypothesis, while the second term is a contraction. By Proposition 1-5, the equation has a unique solution in $w^n L^\infty$, and thus multiplying it by q yields an integrable function.

A similar argument works for $r^{0\pm}$. It is merely necessary to use $ix\,q(x)$ instead of $q(x)$ in the integral equation, which then requires one additional degree of decay from q and its derivatives. □

The integral equations in the spectral variables may be found by calculating $\bar\partial m^0$, i.e., its jump. Both m^{0+} and p^{0+} are holomorphic in the z-variable on the complement of the positive real axis in $\mathbf{C}$. Likewise, m^{0-} and p^{0-} are holomorphic in z off the negative real axis. This is an easy consequence of the holomorphy in z of the contraction $\mathbf{G}^{0\pm}_z \mathbf{q}$ off the positive and negative real axes, respectively. The departure from holomorphy of Green's function $G^{0\pm}_z$ consists entirely of a bounded jump discontinuity across the $\pm$ real z-axis. For such functions, the distributional derivative $\bar\partial = \partial/\partial\bar z$ gives a measure supported along the real z-axis. It is determined by its Radon-Nikodym derivative with respect to Lebesgue measure there, namely by the jump. So, abusing notation only a little, write

$$\bar\partial f(\zeta) \overset{\text{def}}{=} \lim_{\epsilon\to 0^+} f(\zeta + i\epsilon) - f(\zeta - i\epsilon). \tag{2.18}$$

It is easy to compute this quantity from the integral equations above. First, compute the jump in G:

$$\bar\partial G^{0\pm}_\zeta = \begin{cases} i \operatorname{sgn}\zeta\, [e^{ix\zeta} - \chi(\zeta)], & \text{if } \pm\zeta > 0, \\ 0, & \text{if } \pm\zeta \le 0. \end{cases} \tag{2.19}$$

This gives an operator-valued jump:

$$[\bar\partial \mathbf{G}_\zeta^{0\pm}]f(x,\zeta) = \begin{cases} i\ \mathrm{sgn}\,\zeta\ \left[e^{ixz}\int_{\mathbf{R}} e^{-iy\zeta}f(y,\zeta)\,dy - \chi(\zeta)\int_{\mathbf{R}} f(y,\zeta)\,dy\right], & \text{if } \pm\zeta > 0,\\ 0, & \text{if } \pm\zeta \le 0.\end{cases} \tag{2.20}$$

Using these relations, the jumps in both m^{0+} and m^{0-} may be calculated in one step:

$$\begin{aligned}\bar\partial m^{0\pm}(x,\zeta) &= \bar\partial\left(\mathbf{G}_z^{0\pm}\mathbf{q}m^{0\pm}\right)(x,\zeta)\\ &= \left(\bar\partial\mathbf{G}_z^{0\pm}\right)\mathbf{q}m^{0\pm}(x,\zeta+) + \mathbf{G}_{\zeta-}^{0\pm}\mathbf{q}\left(\bar\partial m^{0\pm}(x,\zeta)\right)\\ &= \begin{cases} i\ \mathrm{sgn}\,\zeta\Big[e^{ix\zeta}\displaystyle\int_{\mathbf{R}} e^{-iy\zeta}q(y)m^{0\pm}(y,\zeta+)\,dy & \\ \qquad\qquad - \chi(\zeta)\displaystyle\int_{\mathbf{R}} q(y)m^{0\pm}(y,\zeta+)\,dy\Big] & \text{if } \pm\zeta > 0,\\ \qquad\qquad\qquad + \mathbf{G}_{\zeta-}^{0\pm}\mathbf{q}\bar\partial m^{0\pm}(x,\zeta), & \\ 0, & \text{if } \pm\zeta \le 0.\end{cases}\end{aligned}$$

The integrals are recognizable from the transposition formulas above:

$$\bar\partial m^{0\pm}(x,\zeta) = \begin{cases} i\ \mathrm{sgn}\,\zeta\left[e^{ix\zeta}s^{0\pm}(\zeta) - \chi(\zeta)\alpha^{0\pm}(\zeta+)\right] & \\ \qquad\qquad + \mathbf{G}_{\zeta-}^{0\pm}\mathbf{q}\left(\bar\partial m^{0\pm}(x,\zeta)\right), & \text{if } \pm\zeta > 0,\\ 0, & \text{if } \pm\zeta \le 0.\end{cases}$$

This may be solved by iteration, using Proposition 1-5:

$$\begin{aligned}\bar\partial m^{0\pm}(x,\zeta) &= \begin{cases} i\ \mathrm{sgn}\,\zeta\left(I - \mathbf{G}_{\zeta-}^{0\pm}\mathbf{q}\right)^{-1}\left[e^{ix\zeta}s^{0\pm}(\zeta) - \chi(\zeta)\alpha^{0\pm}(\zeta+)\right], & \text{if } \pm\zeta > 0,\\ 0, & \text{if } \pm\zeta \le 0.\end{cases}\\ &= \begin{cases} i\ \mathrm{sgn}\,\zeta[s^{0\pm}(\zeta)(I - \mathbf{G}_{\zeta-}^{0\pm}\mathbf{q})^{-1}e^{ix\zeta} & \\ \qquad - \chi(\zeta)\alpha^{0\pm}(\zeta+)(I - \mathbf{G}_{\zeta-}^{0\pm}\mathbf{q})^{-1}1], & \text{if } \pm\zeta > 0,\\ 0, & \text{if } \pm\zeta \le 0.\end{cases}\end{aligned}$$

This may be now be written in terms of eigenfunctions and pre-eigenfunctions:

$$\bar\partial m^{0\pm}(x,\zeta) = \begin{cases} i\ \mathrm{sgn}\,\zeta\left[e^{ix\zeta}s^{0\pm}(\zeta)n^{0\pm}(x,\zeta) - \chi(\zeta)\alpha^{0\pm}(\zeta+)m^{0\pm}(x,\zeta-)\right], & \text{if } \pm\zeta > 0,\\ 0, & \text{if } \pm\zeta \le 0.\end{cases} \tag{2.21}$$

The functions $p^{0\pm}$ are also holomorphic off the real z-axis, with a jump discontinuity there. This may be calculated just as for $m^{0\pm}$:

$$\bar\partial p^{0\pm}(x,\zeta) = \begin{cases} i\ \mathrm{sgn}\,\zeta\left[e^{ix\zeta}r^{0\pm}(\zeta)n^{0\pm}(x,\zeta) - \chi(\zeta)\beta^{0\pm}(\zeta+)m^{0\pm}(x,\zeta-)\right], & \text{if } \pm\zeta > 0,\\ 0, & \text{if } \pm\zeta \le 0.\end{cases} \tag{2.22}$$

The improved eigenfunction $n^{0\pm}$ has no holomorphic extension off the real ζ-axis, so it is not possible to compute its departure from holomorphy. However, its derivative with respect to ζ exists and is expressed in terms of the boundary values of the pre-eigenfunctions.

Writing $z=\zeta+i\eta$, denote by D_ζ the derivative in the real direction: $D_\zeta=\partial/\partial\zeta$. Then $D_\zeta n^{0\pm}$ is the total derivative of $n^{0\pm}$ in its second variable. To calculate it, it is useful to first find the commutator of D_ζ with $\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}$:

$$\begin{aligned}
\left[D_\zeta,\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\right]f(x,\zeta) &= \left[D_\zeta(e^{-ix\zeta}G^{0\pm}_{\zeta-i\epsilon})\right]*f(x,\zeta)\\
&= D_\zeta\left[\frac{1}{2\pi}\int_0^{\pm\infty}\frac{e^{ix\xi-ix\zeta}}{\xi-(\zeta-i\epsilon)}\,d\xi-e^{-ix\zeta}l^\pm(\zeta-i\epsilon)\right]*f(x,\zeta)\\
&= D_\zeta\left[\frac{1}{2\pi}\int_{-\zeta}^{\pm\infty}\frac{e^{ix\tau}}{\tau+i\epsilon}\,d\tau-e^{-ix\zeta}l^\pm(\zeta-i\epsilon)\right]*f(x,\zeta)\\
&= \left[\frac{e^{-ix\zeta}}{-2\pi(\zeta-i\epsilon)}+ixe^{-ix\zeta}l^\pm(\zeta-i\epsilon)-e^{-ix\zeta}D_\zeta l^\pm(\zeta-i\epsilon)\right]*f(x,\zeta)\\
&= e^{-ix\zeta}\left(\frac{1}{-2\pi(\zeta-i\epsilon)}-D_\zeta l^\pm(\zeta-i\epsilon)\right)\int_{\mathbf{R}}e^{iy\zeta}f(y,\zeta)\,dy\\
&\qquad+e^{-ix\zeta}l^\pm(\zeta-i\epsilon)\left[ix\int_{\mathbf{R}}e^{iy\zeta}f(y,\zeta)\,dy-\int_{\mathbf{R}}iy\,e^{iy\zeta}f(y,\zeta)\,dy\right].
\end{aligned}$$

Because of its later appearances, introduce one more new fixed function:

$$k^\pm(z)\overset{\text{def}}{=}\left(\frac{1}{-2\pi z}-D_\zeta l^\pm(z)\right),\qquad\text{for } z\in\mathbf{C}\setminus\mathbf{R}^\pm. \tag{2.23}$$

It is easy to show that $k^\pm(z)$ is bounded, analytic, and integrable, in fact with asymptotic behavior

$$k^\pm(z)\to 0 \text{ as } z\to 0,\qquad |k^\pm(z)|=O(|z|^{-2}) \text{ as } |z|\to\infty. \tag{2.24}$$

Then the commutation relation may be written as:

$$\begin{aligned}
\left[D_\zeta,\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\right]f(x,\zeta) &= e^{-ix\zeta}k^\pm(\zeta-i\epsilon)\int_{\mathbf{R}}e^{iy\zeta}f(y,\zeta)\,dy\\
&\qquad+e^{-ix\zeta}l(\zeta-i\epsilon)\left[ix\int_{\mathbf{R}}e^{iy\zeta}f(y,\zeta)\,dy-\int_{\mathbf{R}}iy\,e^{iy\zeta}f(y,\zeta)\,dy\right].
\end{aligned}\tag{2.25}$$

The derivative $D_\zeta n^{0\pm}$ may be calculated in terms of the improved pre-eigenfunctions and scattering data:

$$\begin{aligned}
D_\zeta n^{0\pm}(x,\zeta) &= \lim_{\epsilon\to0^+}D_\zeta(I-\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\mathbf{q})^{-1}1\\
&= \lim_{\epsilon\to0^+}\left[D_\zeta,(I-\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\mathbf{q})^{-1}\right]1\\
&= \lim_{\epsilon\to0^+}(I-\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\mathbf{q})^{-1}\left[D_\zeta,\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\right]\mathbf{q}(I-\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\mathbf{q})^{-1}1\\
&= \lim_{\epsilon\to0^+}(I-\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\mathbf{q})^{-1}\left[D_\zeta,\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\right]\mathbf{q}N^\pm(y,\zeta-i\epsilon,\zeta).
\end{aligned}$$

The commutation relation calculated above breaks the right-hand side into three pieces:

$$(i)\qquad \lim_{\epsilon\to0^+}(I-\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\mathbf{q})^{-1}\left[e^{-ix\zeta}k^\pm(\zeta-i\epsilon)\int_{\mathbf{R}}e^{iy\zeta}q(y)N^\pm(y,\zeta-i\epsilon,\zeta)\,dy\right],$$

$$(ii)\qquad \lim_{\epsilon\to0^+}(I-\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\mathbf{q})^{-1}\left[ixe^{-ix\zeta}\,l^\pm(\zeta-i\epsilon)\int_{\mathbf{R}}e^{iy\zeta}q(y)N^\pm(y,\zeta-i\epsilon,\zeta)\,dy\right],$$

$$(iii)\qquad \lim_{\epsilon\to0^+}(I-\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-i\epsilon}\mathbf{e}\mathbf{q})^{-1}\left[-e^{-ix\zeta}l^\pm(\zeta-i\epsilon)\int_{\mathbf{R}}iy\,e^{iy\zeta}q(y)N^\pm(y,\zeta-i\epsilon,\zeta)\,dy\right].$$

Since $\mathbf{G}^{0\pm}_{\zeta-i\epsilon} \to \mathbf{G}^{0\pm}_{\zeta-}$ in the strong operator topology, and $N^{\pm}(y,\zeta-i\epsilon,\zeta) \to n^{0\pm}(y,\zeta)$ pointwise as $\epsilon \to 0^+$, these limits may be evaluated in terms of the previously defined functions:

$$\begin{aligned}(i) &= e^{-ix\zeta}k^{\pm}(\zeta-)s^{0\mp}(-\zeta)m^{0\pm}(x,\zeta-),\\ (ii) &= e^{-ix\zeta}l^{\pm}(\zeta-)s^{0\mp}(-\zeta)p^{0\pm}(x,\zeta-),\\ (iii) &= -e^{-ix\zeta}l^{\pm}(\zeta-)r^{0\mp}(-\zeta)m^{0\pm}(x,\zeta-).\end{aligned}$$

Putting these parts together relates $D_\zeta n^{0\pm}$ to the pre-eigenfunctions:

$$\begin{aligned}D_\zeta n^{0\pm}(x,\zeta) &= e^{-ix\zeta}k^{\pm}(\zeta-)s^{0\mp}(-\zeta)m^{0\pm}(x,\zeta-)\\ &\quad + e^{-ix\zeta}l^{\pm}(\zeta-)s^{0\mp}(-\zeta)p^{0\pm}(x,\zeta-) - e^{-ix\zeta}l^{\pm}(\zeta-)r^{0\mp}(-\zeta)m^{0\pm}(x,\zeta-).\end{aligned} \tag{2.26}$$

Equations (2.21), (2.22), and (2.26) form a system of distributional differential equations satisfied by the spectral quantities associated to a particular potential q. Compare this with the formal system used in previous analyses of the Benjamin–Ono scattering problem, where the functions involved were ill-behaved due to the singularity of the scattering operator.

Several other properties of the scattering functions and the eigenfunctions and pre-eigenfunctions may be deduced from the integral equations defining them. In particular, the various functions must agree at certain points, which ensures continuity and the convergence of the singular integral operators involved in the inverse scattering transform.

Proposition 2-3. *Given q small with sufficient decay, the functions $m^{0\pm}$, $p^{0\pm}$, $\alpha^{0\pm}$, and $\beta^{0\pm}$ exist and are Hölder continuous in z on $\mathbf{C}\setminus\mathbf{R}^{\pm}$. Furthermore, their nontangential boundary values at $\zeta\pm$, $\zeta\in\mathbf{R}$, are Hölder continuous in ζ of every degree $\delta<1$. Likewise, the functions $s^{0\pm}$, $r^{0\pm}$, and $n^{0\pm}$ are Hölder continuous in $\zeta\in\mathbf{R}$ of every degree $\delta<1$. Finally, these functions satisfy the following relations:*

1. $\lim_{\zeta\to 0^\pm}\alpha^{0\pm}(\zeta-) = s^{0\pm}(0)$.
2. $\lim_{\zeta\to 0^\pm}\beta^{0\pm}(\zeta-) = r^{0\pm}(0)$.
3. $\lim_{\zeta\to 0^\pm} m^{0\pm}(x,\zeta-) = n^{0\pm}(x,0)$ *for every* $x\in\mathbf{R}$.
4. $\lim_{z\to\infty} p^{0\pm}(x,z) = ix$ *for every* $x\in\mathbf{R}$.

Proof. Hölder continuity of $m^{0\pm}$, $n^{0\pm}$, and $p^{0\pm}$ is shown in Proposition 1-7. It holds for $s^{0\pm}$, $r^{0\pm}$, $\alpha^{0\pm}$, and $\beta^{0\pm}$ by dominated convergence. (1) also follows from the dominated convergence theorem. (3) follows from an application of Proposition 1-5 and Proposition 1-6: the function $n^{0\pm}(\zeta) - \mathbf{e}^*m^{0\pm}(\zeta-)$ satisfies the integral equation

$$n^{0\pm} - \mathbf{e}^*m^{0\pm} = 1 - e^{-ix\zeta} + \mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-}\mathbf{eq}(n^{0\pm} - \mathbf{e}^*m^{0\pm}).$$

Now suppose that q is small with sufficient decay. Then $\mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta-}\mathbf{eq}$ is a contraction on $w^\epsilon L^\infty$, uniformly for all ζ. Borrowing a little decay from q, observe that $w^{-\epsilon}(1-e^{-ix\zeta}) \to 0$ in L^∞ as $\zeta\to 0$. Thus, $n^{0\pm} - \mathbf{e}^*m^{0\pm} \to 0$ in $w^\epsilon L^\infty$ as $\zeta \to 0$, proving (3).

If $z\to\infty$ through any region where $|\Im z| > \delta > 0$, then (4) is evident from Eq.(1.14). Since $p^{0\pm}$ is uniformly continuous in $z\in\mathbf{C}\setminus\mathbf{R}^{\pm}$, the result holds in the limit $\delta\to 0$.

Finally, (2) follows from (3) and the dominated convergence theorem. □

3. Recovering the Eigenfunctions from Improved Scattering Data and the Inverse Problem for the Potential

Suppose now that the quantities $\alpha^{0\pm}$, $\beta^{0\pm}$, $s^{0\pm}$, and $r^{0\pm}$ are derived from a particular potential q. Then they are sufficiently well behaved that Eqs.(2.21), (2.22), and (2.26) can be converted into a system of integral equations which determines unique well-behaved functions $m^{0\pm}$, $n^{0\pm}$, and $p^{0\pm}$. Being unique, these coincide with the functions computed directly from q. The solution of the system does not depend upon *a priori* knowledge of q, nor even upon the knowledge that such a q exists. Therefore it provides a key step in the inverse problem for q from the improved scattering data.

As seen above, eigenfunctions and pre-eigenfunctions satisfy relations deducible from the integral equations used to construct them. In particular, the relations must hold for the limit of any inverse process used to reconstruct the potential. Our approach is to write the inverse map as an iteration scheme and force the partial result to satisfy the relations at each step. This also implies that the partial results will be well behaved, that the iterated operators will be bounded, and that the resulting inverse series will converge.

Simpler relations among the original, badly behaved functions may be found in [BC1] and [FA], although the coefficients or scattering data in those simpler relations are singular. The resulting integral equations cannot be solved by iteration, because after even one application the norm estimate of the partial result becomes infinite. Even to show that various quantities in the simpler relations are finite requires subtle cancellation properties equivalent to knowing the potential. As a result, previous existence proofs for BO have used selfadjointness, or else have imposed extremely strong regularity and cancellation properties on both the potential and the scattering data. This last method fails unless the conditions are matched up and one can show that good potentials give good scattering data, and vice versa.

By using the improved eigenfunctions, the relations become more complicated but the coefficients become much more regular. It becomes possible to write down a well-behaved system of equations satisfied by the functions m^0, p^0, and n^0. The coefficients of this system are the well-behaved scattering functions α^0, β^0, s^0, and r^0, as well as the fixed functions χ, l, and k. Solving this system solves an inverse problem for q.

Define two pairs of operators: the jump operators,

$$S^{\pm}(s,\alpha,n,m)(x,\zeta) \overset{\text{def}}{=} \pm i\Big[e^{ix\zeta}s(\zeta)n(x,\zeta) - \chi(\zeta)\alpha(\zeta+)m(x,\zeta-)\Big]\mathbf{1}_{\mathbf{R}^{\pm}}(\zeta), \tag{3.1}$$

and the derivative operators,

$$\begin{aligned} U^{\pm}(s,r,m,p)(x,\zeta) \overset{\text{def}}{=} & k^{\pm}(\zeta-)s(-\zeta)m(x,\zeta-) \\ & + l^{\pm}(\zeta-)s(-\zeta)p(x,\zeta-) - l^{\pm}(\zeta-)r(-\zeta)m(x,\zeta-). \end{aligned} \tag{3.2}$$

Using these, it is simpler to express the jump and derivative relations of the last section:

$$\bar{\partial}m^{0\pm}(x,\zeta) = S^{\pm}\left(s^{0\pm},\alpha^{0\pm},n^{0\pm},m^{0\pm}\right)(x,\zeta), \tag{3.3}$$

$$\bar{\partial}p^{0\pm}(x,\zeta) = S^{\pm}\left(r^{0\pm},\beta^{0\pm},n^{0\pm},m^{0\pm}\right)(x,\zeta), \tag{3.4}$$

$$D_{\zeta}n^{0\pm}(x,\zeta) = U^{\pm}\left(s^{0\mp},r^{0\mp},m^{0\pm},p^{0\pm}\right)(x,\zeta). \tag{3.5}$$

For simplicity, this system will be written only for the functions m^{0+}, n^{0+}, and p^{0+}. Refer to these as m, n, and p.

Define the Cauchy integral C of a Schwartz function f on $\mathbf{R}$ as usual:

$$Cf(z) \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\zeta)\,d\zeta}{\zeta - z}. \tag{3.6}$$

This will be used to construct m and p from their jumps.

Define a particular inverse J of D_u by the integral

$$Jf(\zeta) \overset{\text{def}}{=} \int_0^{\zeta} f(\tau)\,d\tau. \tag{3.7}$$

This will be used to construct n from its derivative.

It is not enough to apply these one-sided inverses to convert the differential relations into integral relations solvable by iteration. It is crucial that the system obtained have a unique solution with the same properties as eigenfunctions coming from a potential. In particular, Proposition 2-3, parts (3) and (4) must hold. To insure this, it is necessary to build the relation into the integral equation.

So observe that the following equations hold for functions related to a small, decreasing potential q:

$$m(x,z) = CS^+(s^+, \alpha^+, n, m)(x,z) - CS^+(s^+, \alpha^+, n, m)(x,0) + n(x,0), \tag{3.8}$$

$$n(x,\zeta) = J\mathbf{e}^* U^+(s^-, r^-, m, p)(x,\zeta) + m(x,0-), \tag{3.9}$$

$$p(x,z) = ix + CS^+(r^+, \beta^+, n, m)(x,z). \tag{3.10}$$

Observe that p is a function of m and n, and that Eq.(3.10) may therefore be eliminated from the system. The two remaining equations define a well-behaved fixed point problem for the pair of functions m and n. This can now be solved by iteration in the metric space appropriate to the equation.

The following general lemma serves to construct the inverse transform:

Lemma 3-1. *Let X and Y be complete metric spaces. Suppose $T : X \times Y \to Y$ is a continuous map on some graph $D = \{(x,y) \in X \times Y : y \in Y_x\}$, where for each $x \in X$, Y_x is a closed subset of Y with $T(x, Y_x) \subset Y_x$. Suppose there is some $C < \infty$ and $\epsilon < 1$ such that for all $(x,y), (x',y') \in D$,*

$$d_Y(T(x,y), T(x',y')) \le C d_X(x,x') + \epsilon d_Y(y,y').$$

Then for each $x \in X$ there is a unique fixed point $y = y(x) \in Y_x$ to the equation $y = T(x,y)$, and the map $x \mapsto y(x)$ is Lipschitz continuous from X to Y with the estimate

$$d_Y(y(x), y(x')) \le \frac{C}{1-\epsilon} d_X(x,x').$$

Proof. Fix x, pick any $y_0 \in Y_x$, and define $y_n \in Y_x$ recursively by $y_n = T(x, y_{n-1})$. Then

$$d_{Y_x}(y_{n+1}, y_n) = d_Y(T(x,y_n), T(x,y_{n-1})) \le \epsilon d_{Y_x}(y_n, y_{n-1}) \le \epsilon^n d_{Y_x}(y_1, y_0).$$

This last term is finite by the continuity of T. Thus, $\epsilon < 1$ implies that $\{y_n\}$ is a Cauchy sequence, which has a limit in Y_x. Since $T(x,\cdot)$ is a contraction on Y_x, this limit is unique, hence it is independent of y_0: call it $y(x)$. To show that $x \mapsto y(x)$ is Lipschitz continuous, let x, x' belong to X. Then

$$d_Y(y(x), y(x')) = d_Y(T(x, y(x)), T(x', y(x'))) \leq C d_X(x, x') + \epsilon d_Y(y(x), y(x'))$$
$$\implies d_Y(y(x), y(x')) \leq \frac{C}{1-\epsilon} d_X(x, x'). \quad \square$$

Naturally, the subtleties of the Benjamin–Ono scattering problem are not evident in this general result but in its nontrivial application. It is necessary to find the right metric spaces for the operators T which are given by the inverse scattering method.

Fix $0 < \delta < 1$ and let W be the space of measurable functions $f : \mathbf{R} \to \mathbf{C}$ such that

(1) There is some $f_\infty \in \mathbf{C}$ for which $f - f_\infty \in L^2(\mathbf{R})$, and
(2) $f \in \mathrm{Lip}_\delta(\mathbf{R})$, i.e., there is some constant $C > 0$ for which $|f(x) - f(x')| \leq C|x - x'|^\delta$. Denote the infimum of all these constants for a given f by $\|f\|_{\mathrm{Lip}_\delta}$.

Evidently, W is a Banach space with norm

$$\|f\|_W = |f_\infty| + \|f - f_\infty\|_{L^2} + \|f\|_{\mathrm{Lip}_\delta},$$

and a complete metric space with metric $d_W(f, g) = \|f - g\|_W$. Define $W^0 = \{f \in W : f_\infty = 0\} = L^2 \cap \mathrm{Lip}_\delta$, and for each $k \in \mathbf{C}$ put $W_k = \{f \in W : f(0) = k\}$. Finally, put $W_k^0 = W^0 \cap W_k$. All of these are closed subsets of W, hence are complete metric subspaces of W.

Multiplication by $\mathbf{1}_{\mathbf{R}^+}$ followed by the Hilbert transform is continuous from W_0^0 to W^0. Note also that W is contained in $L^\infty(\mathbf{R})$, with elementary methods yielding the estimate $\|f\|_{L^\infty} \leq 3\|f\|_W$. An immediate consequence of this is that W is a Banach algebra, with W^0, W_0, and W_0^0 all being (closed) subalgebras.

For $f = (f_1, f_2, f_3, f_4)$, define the closed subset $D \subset W^0 \times W \times W \times W$ to be the graph

$$D = \{f : f_1(0) = f_2(0), f_3(0) = f_4(0)\}. \tag{3.11}$$

Lemma 3-2. *The following estimates hold for all small functions $f, f' \in D$:*

$$\|CS(f) - CS(f')\|_W \leq c_3\|f_1 - f_1'\|_{W^0} + c_4\|f_2 - f_2'\|_W + c_1\|f_3 - f_3'\|_W + c_2\|f_4 - f_4'\|_W. \tag{3.12}$$

Here the constants $c_i, 1 \leq i \leq 4$ are controlled by the corresponding f_i, f_i' and can be made as small as desired:

$$|c_i| = O(\|f_i\| + \|f_i'\|), \qquad \textit{for } i = 1, 2, 3, 4.$$

Proof. For (3.12), it is sufficient to show that S maps D to W_0^0, for then the result follows from the continuity of the Hilbert transform on W_0^0, as stated above. Simplifying the notation somewhat, write

$$\begin{aligned} S(f) - S(f') &= \pm i \left(\mathbf{e}\,[f_1 f_3 - f_1' f_3'] - \chi\,[f_2 f_4 - f_2' f_4']\right) \mathbf{1}_{\mathbf{R}\pm} \\ &= \pm i \left(\mathbf{e}\,[f_1(f_3 - f_3') + f_3'(f_1 - f_1')] - \chi\,[f_2(f_4 - f_4') + f_4'(f_2 - f_2')]\right) \mathbf{1}_{\mathbf{R}\pm}. \end{aligned} \tag{3.13}$$

Now $S(f)$ and $S(f')$ are both in the closed subspace W_0^0, since $f, f' \in D$. This smooths out the discontinuity introduced by $\mathbf{1}_{\mathbf{R}\pm}$. The estimate in Eq.(3.12) thus holds for $S(f) - S(f')$ because W_0^0 is an ideal in the Banach algebra W. The estimate holds for $CS(f) - CS(f')$ because the maximum principle applies, so C is controlled by the Hilbert transform. $\square$

Define $W^1 = \{g \in W(\mathbf{R}^+) : |g(\zeta)| = O(1/\zeta) \text{ as } \zeta \to \infty\}$. This will be shown to be the space of scattering functions s^0, r^0 arising from potentials q with one derivative. Define also jW^1, where $j(\zeta) \stackrel{\text{def}}{=} \max\{|l^\pm(\zeta\pm)|, |k^\pm(\zeta\pm)|\}$. This j is introduced because U involves multiplication by l and k. Eqs. (2.17) and (1.2) imply that $j(\zeta) \sim 1/\zeta$ as $\zeta \to \infty$ and that $j(\zeta) \sim -\log\zeta$ as $\zeta \to 0$. In particular, if $g \in jW^1$, then $g(\zeta) = O(1/\zeta^2)$ as $\zeta \to \infty$, and $g(\zeta) = O(|\log\zeta|)$ as $\zeta \to 0$. Both W^1 and jW^1 are Banach spaces with the obvious norms. In addition, W^1 is a Banach subalgebra (in fact an ideal) of W.

Put $\mathbf{W} = W^1 \times W^1 \times W \times W$. Then a result similar to Lemma 3-2 is true for $J\mathbf{e}^*U$ on $\mathbf{W}$.

Lemma 3-3. *The following estimates hold for small $f, f' \in \mathbf{W}$:*

$$\|J\mathbf{e}^*U(f) - J\mathbf{e}^*U(f')\|_{W^1} \leq b_{34}\|f_1 - f_1'\|_{W^1} + b_3\|f_2 - f_2'\|_{W^1} + b_{12}\|f_3 - f_3'\|_W + b_1\|f_4 - f_4'\|_W. \tag{3.14}$$

The constants $b_I, I = 1, 3, \{1,2\}, \{3,4\}$ are controlled by the corresponding $\{f_i, f_i' : i \in I\}$ and can be made as small as desired:

$$|b_I| = O(\sum_{i\in I} [\|f_i\| + \|f_i'\|]), \qquad \text{for } I = 1, 3, \{1,2\}, \{3,4\}.$$

Proof. Note first that $U : \mathbf{W} \to jW^1$ is continuous: multiplication by $s^{0\pm}$ or $r^{0\pm}$ maps W to W^1, and then multiplication by $k^\pm$ or $l^\pm$ maps W^1 to jW^1.

Second, observe that $\mathbf{e}^* : jW^1 \to jW^1$ is continuous for any fixed $x \in \mathbf{R}$.

Third, note that $jW^1 \subset L^1(\mathbf{R}^+)$, so that $J\mathbf{e}^*U$ is bounded and absolutely continuous. It tends to a constant at infinity at a rate that can be estimated by Schwarz' inequality: the difference between them is dominated by $O(1/\zeta)$ as $\zeta \to \infty$. In particular, this is in $L^2(\mathbf{R}^+)$. Also, since $f \in jW^1$ is bounded for $\zeta > 1$ and $O(1 - \log\zeta)$ for $0 < \zeta \leq 1$, Jf satisfies a uniform Hölder estimate for every fixed degree $\delta < 1$. Together, these show that $J : jW^1 \to W^1$ is continuous.

To show that the estimate holds, consider the decomposition

$$\begin{aligned} U(f) - U(f') &= k\,(f_1 f_3 - f_1' f_3') \\ &\quad - l\,(f_2 f_3 - f_2' f_3') + l\,(f_1 f_4 - f_1' f_4') \\ &= k\,(f_1[f_3 - f_3'] + f_3'[f_1 - f_1']) \\ &\quad + l\,(f_1[f_4 - f_4'] + f_4'[f_1 - f_1'] - l\,(f_2[f_3 - f_3'] + f_3'[f_2 - f_2'])) . \end{aligned} \tag{3.15}$$

Since the components of $\mathbf{W}$ are Banach algebras, $\|U(f) - U(f')\|_{jW^1}$ is controlled by the expression

$$b_{34}\|f_1 - f_1'\|_{W^1} + b_3\|f_2 - f_2'\|_{W^1} + b_{12}\|f_3 - f_3'\|_W + b_1\|f_4 - f_4'\|_W. \tag{3.16}$$

Then $J\mathbf{e}^*U$ is controlled by the expression in (3.16), completing the proof. □

Corollary 3-4. *The map $(m, n) \mapsto p$ defined by Eq.(3.10) is Lipschitz continuous in $m, n \in W$. The Lipschitz constant is $O(\|r\|_{W^0} + \|\beta\|_W)$.*

Proof. Apply Lemma 3-2 with $f_1 = r, f_2 = \beta, f_3 = n, f_4 = m$. □

Now write $p = p(m, n)$, where $(m, n) \mapsto p$ is Lipschitz with small constant. Consider the map $(m, n) \mapsto T(s, r, \alpha, \beta, m, n)$, where $T = (T_1, T_2)$ is defined by:

$$T_1(s, r, \alpha, \beta, m, n) = CS^+(s^+, \alpha^+, n, m)(x, z) - CS^+(s^+, \alpha^+, n, m)(x, 0) + n(x, 0), \tag{3.17}$$

$$T_2(s, r, \alpha, \beta, m, n) = J\mathbf{e}^*U^+(s^-, r^-, m, p(m, n))(x, \zeta) + m(x, 0-). \tag{3.18}$$

It is now a simple matter to prove the main theorem on the inverse problem for the improved eigenfunctions and pre-eigenfunctions from the improved scattering data:

Theorem 3-5. *For small s, r, α, β in the subset $X \subset W^1 \times W^1 \times W \times W$ defined by $\{\alpha(0) = s(0), \beta(0) = r(0)\}$, the fixed-point problem*

$$(m,n) = T(s,r,\alpha,\beta,m,n)$$

has a unique solution $(m,n) \in W \times W$. Furthermore, the implicit map $(s,r,\alpha,\beta) \mapsto (m,n,p(m,n))$ is Lipschitz continuous in all components.

Proof. Let X be as defined, let $Y = W \times W$, and set $D = X' \times Y$ for a sufficiently small $X' \subset X$. Then T satisfies the hypothesis of Lemma 3-1 with $(m,n) \in Y$, proving most of the result. The Lipschitz continuity of $(s,r,\alpha,\beta) \mapsto p$ follows from Corollary 3-4. □

Having recovered the eigenfunctions from the improved scattering data, there are several representations of the potential function q available. The most direct is the Born approximation.

Suppose that q has enough decay to guarantee the existence of the various eigenfunctions and scattering data, and is smooth enough so that the scattering functions $s^{0\pm}$ decrease to 0 at infinity. Then q may be computed from the asymptotic behavior of those eigenfunctions. This behavior is determined by the integral equations the eigenfunctions solve.

Eq.(1.12) gives the asymptotic behavior of $n^{0\pm}(x,\zeta)$ as $\zeta \to \pm\infty$:

$$\begin{aligned} n^{0\pm}(x,\zeta) &= 1 + \mathbf{e}^* \mathbf{G}^{0\pm}_{\zeta-} \mathbf{e} \mathbf{q} n^{0\pm}(x,\zeta) \\ &= 1 + \lim_{\epsilon \to 0^+} \left(\int_{-\zeta}^{\pm\infty} \frac{e^{ixt}}{t+i\epsilon}\, dt \right) * \mathbf{q} n^{0\pm}(x,\zeta) \\ &\quad - e^{-ix\zeta} l(\zeta-) \int_{\mathbf{R}} e^{iy\zeta} q(y) n^{0\pm}(y,\zeta)\, dy. \end{aligned} \tag{3.19}$$

Now, the last integral in this equation is bounded uniformly in ζ, while $l(\zeta-) \to 0$ as $\zeta \to \pm\infty$. In addition, the x-derivative of the last right-hand term vanishes as $z \to \infty$, so that the integral equation for $n^{0+}(x,\infty)$ may be reduced as follows:

$$n^{0+}(x,\infty) = 1 + \lim_{\epsilon \to 0^+} \left(\int_{-\infty}^{\infty} \frac{e^{ixt}}{t+i\epsilon}\, dt \right) * \mathbf{q} n^{0+}(x,\infty),$$

$$\implies \frac{1}{i}\frac{d}{dx} n^{0+}(x,\infty) = q(x) n^{0+}(x,\infty),$$

$$\implies n^{0+}(x,\infty) = c \, \exp\left[i \int_{-\infty}^{x} q(y)\, dy \right], \tag{3.20}$$

where $c = n^{0+}(-\infty,\infty)$. Alternatively, take the other primitive of $q(x)$ and write:

$$n^{0+}(x,\infty) = c' \, \exp\left[-i \int_{x}^{-\infty} q(y)\, dy \right]. \tag{3.20$'$}$$

Here $c' = n^{0+}(\infty,\infty)$. Notice that the equation for $n^{0-}(x,-\infty)$ is very similar:

$$n^{0-}(x,-\infty) = 1 + \left(\int_{\infty}^{-\infty} \frac{e^{ixt}}{t+i\epsilon}\, dt \right) * \mathbf{q} n^{0-}(x,-\infty),$$

$$\implies \frac{1}{i}\frac{d}{dx} n^{0-}(x,-\infty) = -q(x) n^{0-}(x,-\infty),$$

$$\implies n^{0-}(x,-\infty) = \tilde{c} \, \exp\left[-i \int_{-\infty}^{x} q(y)\, dy \right], \tag{3.21}$$

$$\implies n^{0-}(x,-\infty) = \tilde{c}' \, \exp\left[i \int_{x}^{\infty} q(y)\, dy \right]. \tag{3.21$'$}$$

Observe that $n^{0\pm}(x,\pm\infty)$ may have the same value at $x=+\infty$ and $x=-\infty$ if and only if $\int_{\mathbf{R}} q = 0$.

In particular, these equations for $n^{0\pm}$ may be used in a Born approximation for q:

$$q(x) = \lim_{\zeta\to\pm\infty} \frac{1}{i}\frac{d}{dx}\log n^{0\pm}(x,\zeta). \tag{3.22}$$

Either of the $^{\pm}$ parts of $n^{0\pm}$ contains enough information to recover the potential q. Unfortunately, this method does not lend itself to useful estimates, nor to efficient numerical algorithms.

There are also integral representation of q. Recall that P^+ and P^- denote the orthogonal projections of $L^2(\mathbf{R})$ onto H^2 and $\overline{H^2}$, respectively. Then

$$q(x) = P^+q(x) + P^-q(x). \tag{3.23}$$

The equation defining n may be rewritten as

$$\left(\frac{1}{i}\frac{d}{dx} - \zeta\right)\mathbf{e}n^+(x,\zeta) = P^+\mathbf{q}\mathbf{e}n^+(x,\zeta), \qquad \text{if } \zeta>0, \tag{3.24+}$$

$$\left(\frac{1}{i}\frac{d}{dx} - \zeta\right)\mathbf{e}n^-(x,\zeta) = -P^-\mathbf{q}\mathbf{e}n^-(x,\zeta), \qquad \text{if } \zeta<0. \tag{3.24−}$$

Likewise, the equation defining $m^\pm$ may be rewritten as

$$\frac{1}{i}\frac{d}{dx}m^\pm(x,\zeta\pm) - \zeta\left[m^\pm(x,\zeta\pm)-1\right] = \begin{cases} P^+\mathbf{q}m^+(x,\zeta\pm), & \text{if } \zeta>0,\\ -P^-\mathbf{q}m^-(x,\zeta\pm), & \text{if } \zeta<0.\end{cases} \tag{3.25}$$

The jump in $m^\pm$ expresses $m^\pm$ in terms of $s^\pm$ and $n^\pm$. Using the Cauchy integral operator C gives the following representation:

$$m^\pm(x,z) = 1 + C[i\,\mathrm{sgn}\,\zeta\, e^{ix\zeta}s^\pm(\zeta)n^\pm(x,\zeta)\mathbf{1}_{\mathbf{R}\pm}(\zeta)](z) \stackrel{\text{def}}{=} 1 + C\mathbf{s}\mathbf{e}n^\pm(x,z), \tag{3.26}$$

where $\mathbf{s}$ denotes the operator multiplying by $+i\,s^+(\zeta)\mathbf{1}_{\mathbf{R}+}(\zeta)$, if $\zeta>0$, or $-i\,s^-(\zeta)\mathbf{1}_{\mathbf{R}-}(\zeta)$, if $\zeta<0$.

Rewriting Eq.(3.25) in terms of $m-1$ yields:

$$P^+q(x) = \left(\frac{1}{i}\frac{d}{dx} - P^+\mathbf{q} - \zeta\right)[m^+(x,\zeta\pm)-1], \qquad \text{if } \zeta>0, \tag{3.27+}$$

$$P^-q(x) = -\left(\frac{1}{i}\frac{d}{dx} - P^-\mathbf{q} - \zeta\right)[m^-(x,\zeta\pm)-1], \qquad \text{if } \zeta<0. \tag{3.27−}$$

Using Eq.(3.26) reduces this to:

(3.28+)

$$\begin{aligned}
P^+q(x) &= \left(\frac{1}{i}\frac{d}{dx} - P^+\mathbf{q} - \zeta\right)C\mathbf{s}\mathbf{e}n^+(x,\zeta), \qquad \text{if } \zeta>0,\\
&= \left[\left(\frac{1}{i}\frac{d}{dx} - P^+\mathbf{q} - \zeta\right), C\mathbf{s}\right]\mathbf{e}n^+(x,\zeta),\\
&= [-\zeta, C]\,\mathbf{s}\mathbf{e}n^+(x,\zeta), \qquad \text{since } \frac{1}{i}\frac{d}{dx} \text{ and } P^+\mathbf{q} \text{ commute with } C \text{ and } \mathbf{s},\\
&= \frac{1}{2\pi}\int_0^\infty \mathrm{sgn}\,\zeta\, e^{ix\zeta}s^+(\zeta)n^+(x,\zeta)\,d\zeta,
\end{aligned}$$

(3.28−)

$$\begin{aligned}P^-q(x) &= -\left(\frac{1}{i}\frac{d}{dx} - P^-\mathbf{q} - \zeta\right) C\mathbf{s}\mathbf{e}n^-(x,\zeta), \qquad \text{if } \zeta < 0,\\ &= \left[-\left(\frac{1}{i}\frac{d}{dx} - P^-\mathbf{q} - \zeta\right), C\mathbf{s}\right]\mathbf{e}n^-(x,\zeta),\\ &= [\zeta, C]\,\mathbf{s}\mathbf{e}n^-(x,\zeta), \qquad \text{since } \frac{1}{i}\frac{d}{dx} \text{ and } P^-\mathbf{q} \text{ commute with } C \text{ and } \mathbf{s},\\ &= -\frac{1}{2\pi}\int_{-\infty}^{0} \operatorname{sgn}\zeta\, e^{ix\zeta} s^-(\zeta) n^-(x,\zeta)\,d\zeta.\end{aligned}$$

If it is understood that $\mathbf{n}(x,\zeta) = n^{\pm}(x,\zeta)$ if $\pm\zeta > 0$, then these can be combined to give a representation for q in terms of the scattering function $\mathbf{s}$:

$$q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\zeta}\mathbf{s}\mathbf{n}(x,\zeta)\,d\zeta \tag{3.29}$$

Observe that this formula agrees with the abstract result from spectral theory, where $s^{\pm}$ is the distorted Fourier transform of $P^{\pm}q$, and the transformation is given in terms of the eigenfunctions $n^{\pm}$. Unfortunately, this result is useless without reasonable estimates on $n^{\pm}$ and $s^{\pm}$ near $z = 0$, which requires the use of $n^{0\pm}$ and $s^{0\pm}$.

Equations (2.11) to (2.17) may be used to give representations of q in terms of good eigenfunctions:

(3.30)

$$\begin{aligned}q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\zeta}&\frac{s^{0\pm}(\zeta)}{1 - l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)}\\ &\times\left[n^{0\pm}(x,\zeta) + e^{-ix\zeta} l^{\pm}(\zeta+) s^{0\mp}(-\zeta)\frac{m^{0\pm}(x,\zeta+)}{1 - l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)}\right] d\zeta\\ = \frac{1}{2\pi}\int_{-\infty}^{\infty}&\left[e^{ix\zeta}\frac{s^{0\pm}(\zeta)n^{0\pm}(x,\zeta)}{1 - l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)} + \frac{l^{\pm}(\zeta+)s^{0\pm}(\zeta)s^{0\mp}(-\zeta)m^{0\pm}(x,\zeta+)}{[1 - l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)]^2}\right] d\zeta.\end{aligned}$$

4. Relations and Dependencies Among the Scattering Functions

To recover the eigenfunctions, and thus the potential q, from all the various functions used above is not too difficult. What is surprising is that the single function r^0 suffices to determine all the others and therefore to determine q. As a consequence, solving the initial value problem for r^0 solves the Benjamin–Ono initial value problem.

Although there are 4 functions (with $\pm$ parts) used as scattering data, they are all related by virtue of coming from a single potential q. These relations may be stated abstractly, without reference to q, using the jump and derivative calculations below. This reduces the redundancy in the scattering data prior to solving the inverse problem for the potential.

The functions $\alpha^{0\pm}$ and $\beta^{0\pm}$ are holomorphic off the real z-axis, so they are determined by their jumps across the real axis. These jumps may be computed in terms of the jumps in $m^{0\pm}$ and $p^{0\pm}$, which involve the function $n^{0\pm}$ as well.

First, calculate the jump in $\alpha^{0\pm}$ across the real axis:

$$
\begin{aligned}
\bar\partial\alpha^{0\pm}(\zeta) &= \int_{\mathbf{R}} q(x)\bar\partial m^{0\pm}(x,\zeta)\,dx \\
&= \begin{cases} i\,\mathrm{sgn}\,\zeta\left[s^{0\pm}(\zeta)s^{0\mp}(-\zeta)-\chi(\zeta)\alpha^{0\pm}(\zeta+)\alpha^{0\pm}(\zeta-)\right], & \text{if } \pm\zeta>0,\\ 0, & \text{if } \pm\zeta\le 0.\end{cases}
\end{aligned}
\tag{4.1}
$$

This may also be expressed in terms of the jump operator defined in Eq. (3.1):

$$
\bar\partial\alpha^{0\pm}(\zeta) = S^{\pm}(s^{0\pm},\alpha^{0\pm}_{+},\check s^{0\mp},\alpha^{0\pm}_{-})(0,\zeta).
\tag{4.2}
$$

Here the subscripts indicate whether to use the upper or lower boundary values at $\mathbf{R}$ of a function defined on $\mathbf{C}\setminus\mathbf{R}$. Also, $\check s^{0\mp}(\zeta)\stackrel{\text{def}}{=} s^{0\mp}(-\zeta)$. This function satisfies exactly the same estimates as $s^{0\mp}$.

For future use, we calculate the jump in $1-l^{\pm}\alpha^{0\pm}$:

$$
\begin{aligned}
&l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)-l^{\pm}(\zeta-)\alpha^{0\pm}(\zeta-)= \\
&\quad=\left[l^{\pm}(\zeta+)-l^{\pm}(\zeta-)\right]\alpha^{0\pm}(\zeta+)+l^{\pm}(\zeta-)\left[\alpha^{0\pm}(\zeta+)-\alpha^{0\pm}(\zeta-)\right]\\
&\quad= i\,\mathrm{sgn}\,\zeta\,\chi(\zeta)\alpha^{0\pm}(\zeta+)+i\,\mathrm{sgn}\,\zeta\left[s^{0\pm}(\zeta)s^{0\mp}(-\zeta)-\chi(\zeta)\alpha^{0\pm}(\zeta+)\alpha^{0\pm}(\zeta-)\right]l^{\pm}(\zeta-),
\end{aligned}
$$

$$
\begin{aligned}
\implies 1-l^{\pm}(\zeta-)\alpha^{0\pm}(\zeta-) &= 1-l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)\\
&\quad+i\,\mathrm{sgn}\,\zeta\,\left[\chi(\zeta)\alpha^{0\pm}(\zeta+)-\chi(\zeta)\alpha^{0\pm}(\zeta+)\alpha^{0\pm}(\zeta-)+l^{\pm}(\zeta-)s^{0\pm}(\zeta)s^{0\mp}(-\zeta)\right],
\end{aligned}
$$

$$
\begin{aligned}
\implies &\left[1-l^{\pm}(\zeta-)\alpha^{0\pm}(\zeta-)\right]\left[1-i\,\mathrm{sgn}\,\zeta\,\chi(\zeta)\alpha^{0\pm}(\zeta+)\right]=\\
&\qquad\qquad= 1-l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)+i\,\mathrm{sgn}\,\zeta\,l^{\pm}(\zeta-)s^{0\pm}(\zeta)s^{0\mp}(-\zeta).
\end{aligned}
$$

Dividing by $1-l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)$ gives:

$$
\frac{i\,\mathrm{sgn}\,\zeta l^{\pm}(\zeta-)s^{0\pm}(\zeta)s^{0\mp}(-\zeta)}{1-l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)}=\frac{1-l^{\pm}(\zeta-)\alpha^{0\pm}(\zeta-)}{1-l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)}\left[1-i\,\mathrm{sgn}\,\zeta\,\chi(\zeta)\alpha^{0\pm}(\zeta+)\right]-1.
\tag{4.3}
$$

This equation holds even for complex q.

Similarly, compute the jump in $\beta^{0\pm}$ across the real axis:

$$
\begin{aligned}
\bar\partial\beta^{0\pm}(\zeta) &= \int_{\mathbf{R}} q(x)\bar\partial p^{0\pm}(x,\zeta)\,dx \\
&= \begin{cases} i\,\mathrm{sgn}\,\zeta\left[r^{0\pm}(\zeta)s^{0\mp}(-\zeta)-\chi(\zeta)\beta^{0\pm}(\zeta+)\alpha^{0\pm}(\zeta-)\right], & \text{if } \pm\zeta>0,\\ 0, & \text{if } \pm\zeta\le 0.\end{cases}
\end{aligned}
\tag{4.4}
$$

Using the same subscript convention as in (4.2), this may also be expressed in terms of the jump operator $S^{\pm}$:

$$
\bar\partial\beta^{0\pm}(\zeta) = S^{\pm}(r^{0\pm},\beta^{0\pm}_{+},\check s^{0\mp},\alpha^{0\pm}_{-})(0,\zeta).
\tag{4.5}
$$

On the other hand, there is no jump in $s^{0\pm}$. This may be related to the other scattering data by differentiation with respect to ζ:

$$
\begin{aligned}
D_\zeta s^{0\pm}(\zeta) &= D_\zeta \int_{\mathbf{R}} e^{-ix\zeta} q(x) n^{0\mp}(x,-\zeta)\,dx \\
&= \int_{\mathbf{R}} -ix\, e^{-ix\zeta} q(x) n^{0\mp}(x,-\zeta)\,dx + \int_{\mathbf{R}} e^{-ix\zeta} q(x) D_u n^{0\mp}(x,-\zeta)\,dx \\
&= -r^{0\pm}(\zeta) \\
&\quad - k^{\mp}((-\zeta)-) s^{0\pm}(\zeta) \alpha^{0\mp}((-\zeta)-) \\
&\quad - l^{\mp}((-\zeta)-) s^{0\pm}(\zeta) \beta^{0\mp}((-\zeta)-) \\
&\quad + l^{\mp}((-\zeta)-) r^{0\pm}(\zeta) \alpha^{0\mp}((-\zeta)-).
\end{aligned} \tag{4.6}
$$

In turn, this may be expressed as the derivative operator $U^{\pm}$ defined in Eq. (3.2):

$$
D_\zeta s^{0\pm}(\zeta) = -r^{0\pm}(\zeta) - U^{\mp}(\check{s}^{0\pm}, \check{r}^{0\pm}, \alpha_-^{0\mp}, \beta_-^{0\mp})(0,-\zeta). \tag{4.7}
$$

Here $\check{r}^{0\pm}(-\zeta) \overset{\text{def}}{=} r^{0\pm}(\zeta)$.

The scattering data are determined by these jumps and derivatives. The Cauchy integral C gives the upper and lower nontangential boundary values for functions analytic off $\mathbf{R}$, from their jumps. Namely, if the jump of $F(z)$ is the Schwartz function $f(\zeta)$, then for some constant F_∞,

$$
F(\zeta\pm) - F_\infty = Cf(\zeta\pm) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\tau)\,d\tau}{\tau - (\zeta \pm i\epsilon)}, \tag{4.8}
$$

These operators are to be distinguished from P^+ and P^- because they act in the ζ-variable, rather than in the x-variable. The constant F_∞ is determined by Proposition 2-3. Thus, define the operator $T = (T_+, T_-)$ by

$$
\begin{aligned}
T_\pm(s^{0\pm}, \alpha_+^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm})(\zeta) &= s^{0\pm}(0) \\
&\quad + CS^{\pm}(s^{0\pm}, \alpha_+^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm})(0,\zeta\pm) - CS^{\pm}(s^{0\pm}, \alpha_+^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm})(0,0\pm).
\end{aligned} \tag{4.9}
$$

Then scattering data from a small potential q satisfies the equation

$$
(\alpha_+^{0\pm}, \alpha_-^{0\pm}) = T(s^{0\pm}, \alpha_+^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm}).
$$

But if $s^{0\pm} \in W^0$ is small, and if T is restricted to the graph

$$
D = \{s^{0\pm}(0) = \alpha^{0\pm}(0+),\ \check{s}^{0\mp}(0) = \alpha^{0\pm}(0-)\} \subset W^0 \times W \times W \times W,
$$

then this operator T satisfies the hypothesis of Lemma 3-1 exactly as in the proof of Lemma 3-2. Thus the fixed point $(\alpha_+^{0\pm}, \alpha_-^{0\pm})$ is uniquely determined and depends in a Lipschitz continuous manner on $s^{0\pm}$. But then the holomorphic function $\alpha^{0\pm}$ is likewise determined. This reduces the number of scattering functions from 4 pairs (of $^{\pm}$ functions) to 3.

In fact, only one side of each function $s^{0\pm}$ is used: $s^{0+}(\zeta)$ if $\zeta > 0$, and $s^{0-}(\zeta)$ if $\zeta < 0$. In a sense, these two half functions are really just a single scattering datum. The implicit dependence of $\alpha^{0\pm}$ on these may be stated as the following:

Lemma 4-1. *Suppose that q is a small potential with associated scattering data $\alpha^{0\pm}$, and $s^{0\pm}$. If $s^{0+}(\zeta), \zeta > 0$ and $s^{0-}(\zeta), \zeta < 0$ are small functions in $W^0(\mathbf{R}^+)$ and $W^0(\mathbf{R}^-)$, respectively, then there is a Lipschitz continuous map $A = (A_+, A_-)$, $A : W^0 \times W^0 \to W \times W$ of small constant such that $(\alpha_+^{0\pm}, \alpha_-^{0\pm}) = A(s^{0\pm}, \check{s}^{0\mp})$. These "boundary values" determine the holomorphic extension $\alpha^{0\pm}(z) = A^\pm(s^{0\pm}, \check{s}^{0\mp})(z)$ to $\mathbf{C} \setminus \mathbf{R}^\pm$ by the formula*

$$A^\pm(z) = C([A_+ - A_-]\mathbf{1}_{\mathbf{R}^\pm})(z) + C_\infty, \tag{4.10}$$

where $C_\infty \stackrel{\text{def}}{=} -A_-(s^{0\pm}, \check{s}^{0\mp})(0) + s^{0\pm}(0)$ is chosen to satisfy Proposition 2-3. □

A similar result holds for the function $\beta^{0\pm}$. First the upper boundary value $\beta_+^{0\pm}$ is calculated by iteration, then the lower boundary value $\beta_-^{0\pm}$ is obtained from the jump relation:

Lemma 4-2. *Suppose that q is a small potential with associated scattering data $\alpha^{0\pm}$, $\beta^{0\pm}$, $r^{0\pm}$, and $s^{0\pm}$. If $s^{0+}(\zeta), \zeta > 0$ and $s^{0-}(\zeta), \zeta < 0$ are small functions in $W^0(\mathbf{R}^\pm)$, and $r^{0+}(\zeta), \zeta > 0$ and $r^{0-}(\zeta), \zeta < 0$ are small functions in $W^0(\mathbf{R}^\pm)$, and $\alpha^{0\pm}(\zeta\pm)$ are small functions in W, then there is a Lipschitz continuous map $B = (B_+, B_-)$ of small constant, where $B_\pm : W^0 \times W^0 \times W \to W$, such that $(\beta_+^{0\pm}, \beta_-^{0\pm}) = B(r^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm})$. These determine the function $\beta^{0\pm} = B^\pm(r^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm})$, holomorphic on $\mathbf{C} \setminus \mathbf{R}^\pm$, by the formula*

$$B^\pm(z) = C([B_+ - B_-]\mathbf{1}_{\mathbf{R}^\pm})(z) + B_\infty, \tag{4.11}$$

where $B_\infty \stackrel{\text{def}}{=} -B_-(r^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm})(0) + r^{0\pm}(0)$ is chosen to satisfy Proposition 2-3.

Proof. Set $T = T_+$ as in Eq.(4.9), and consider the fixed point problem that $\beta_+^{0\pm}$ solves by virtue of Eq.(4.5):

$$\beta_+^{0\pm} = T_+(r^{0\pm}, \beta_+^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm}).$$

For α^0, s^0, r^0 as given, the operator satisfies the hypothesis of Lemma 3-1 for $\beta_+^{0\pm} \in W$. Thus the boundary value $\beta_+^{0\pm}$ is uniquely determined as an implicit Lipschitz function of the other three variables. Write this as $\beta_+^{0\pm} = B_+(r^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm})$.

B_- can be calculated from the jump relation:

$$\begin{aligned}\beta_-^{0\pm}(\zeta) &= \beta_+^{0\pm}(\zeta) - \bar\partial\beta^{0\pm}(\zeta)\\ &= \beta_+^{0\pm}(\zeta) - S^\pm(r^{0\pm}, \beta_+^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm})(0, \zeta).\end{aligned}$$

Into this, B_+ may be substituted in two places:

$$\begin{aligned}\beta_-^{0\pm}(\zeta) &= B_+(r^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm})(\zeta) - S^\pm(r^{0\pm}, B_+(r^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm}), \check{s}^{0\mp}, \alpha_-^{0\pm})(0, \zeta)\\ &\stackrel{\text{def}}{=} B_-(r^{0\pm}, \check{s}^{0\mp}, \alpha_-^{0\pm})(\zeta).\end{aligned} \tag{4.12}$$

This map is Lipschitz continuous with small constant whenever $S^\pm$ and B_+ are. □

Finally, $s^{0\pm}$ can be related to $r^{0\pm}, \alpha^{0\pm}, \beta^{0\pm}$. The particular antiderivative J defined in Eq.(3.7) inverts D_ζ in Eq.(4.7), up to a constant determined by Proposition 2-3.

Define yet another space:

$$W^2 = W^2(\mathbf{R}^\pm) \stackrel{\text{def}}{=} \{f \in W(\mathbf{R}^\pm) : f(\zeta) = O(1/\zeta^2) \text{ as } \zeta \to \pm\infty\}. \tag{4.13}$$

This space contains scattering data from potentials q with two derivatives. Note that W^2 may be used instead of jW^1 in Lemma 3-3.

Lemma 4-3. *Suppose that q is a small potential with associated scattering data $\alpha^{0\pm}$, $\beta^{0\pm}$, $r^{0\pm}$, and $s^{0\pm}$. If $r^{0+}(\zeta), \zeta > 0$ and $r^{0-}(\zeta), \zeta < 0$ are small functions in $W^2(\mathbf{R}^{\pm})$, and $\beta_{\pm}^{0\pm}, \alpha_{\pm}^{0\pm}$ are small functions in W, then there is a Lipschitz continuous map $R : W^2 \times W \times W \to W^1$ of small constant such that*

$$s^{0\pm} = R(r^{0\pm}, \alpha^{0\pm}, \beta^{0\pm}).$$

Proof. Solving Eq.(4.7) with J in order to guarantee that $s^{0\pm} \in W^1$ gives the equation

$$s^{0\pm}(\zeta) = -J(r^{0\pm})(\zeta) - J\check{U}^{\mp}(\check{s}^{0\pm}, \check{r}^{0\pm}, \alpha_{-}^{0\mp}, \beta_{-}^{0\mp})(\zeta) + J_{\pm\infty}. \tag{4.14}$$

Here $J_{\pm\infty}$ is shorthand for the (finite) limit as $\zeta \to \pm\infty$ of the other two terms of the right-hand side. Also, $\check{U}^{\pm}(0,\zeta) \stackrel{\text{def}}{=} U^{\mp}(0,-\zeta)$. This satisfies the same estimates as $U^{\pm}$.

With r^0, α^0, β^0 as given, the right-hand side belongs to $W^1(\mathbf{R}^{\pm})$ for small $s^{0\pm} \in W^1(\mathbf{R}^{\pm})$. Hölder continuity is clear for $J(r^{0\pm})$, and follows for the second and third terms by part of Lemma 3-3. Another part of that lemma shows that the differences between the first two terms and their finite limits at $\pm\infty$ vanish like $1/|\zeta|$. A third part of Lemma 3-3 shows that the right-hand side is a Lipschitz map in s^0 with a constant controlled by $\|r^0\|_{W^2} + \|\alpha^0\|_W + \|\beta^0\|_W$. If this is small, Lemma 3-1 applies to give the result. $\square$

Substituting for $\alpha_{-}^{0\pm}, \beta_{-}^{0\pm}$ from Lemmas 4-1 and 4-2 into the formula from Lemma 4-3 gives a fixed point problem for $s^{0\pm}$ involving only $r^{0\pm}$. The solvability of this problem eliminates all but one of the scattering functions and is the main result of this section:

Theorem 4-4. *If $r^{0\pm}$ are small functions in $W^2(\mathbf{R}^{\pm})$, which are associated to a potential q by the relation in Eq.(2.4), then the other scattering data $\alpha^{0\pm}$, $\beta^{0\pm}$, and $s^{0\pm}$ related to that potential may be written in terms of $r^{0\pm}$:*

$$\alpha^{0\pm} = \mathcal{A}(r^{0\pm}), \qquad \beta^{0\pm} = \mathcal{B}(r^{0\pm}), \qquad s^{0\pm} = \mathcal{S}(r^{0\pm}).$$

The functions $\mathcal{A} : W^2 \to W$, $\mathcal{B} : W^2 \to W$, and $\mathcal{S} : W^2 \to W^1$ are Lipschitz continuous.

Proof. Substitution into Eq.(4.14) from Lemmas 4-1 and 4-2 yields

$$\begin{aligned} s^{0\pm}(\zeta) = {} & J_{\pm\infty} - J(r^{0\pm})(\zeta) \\ & - J\check{U}^{\mp}\left(\check{s}^{0\pm}, \check{r}^{0\pm}, A_{-}(s^{0\mp}, \check{s}^{0\pm}), B_{-}(r^{0\mp}, s^{0\pm}, A_{-}(s^{0\mp}, \check{s}^{\pm})\right)(\zeta). \end{aligned} \tag{4.15}$$

The right-hand side is Lipschitz from $(r^{0\pm}, s^{0\pm}) \in W^2 \times W^1$ to W^1, by composition. For $r^{0\pm}$ small, it is a contraction in $s^{0\pm}$. Applying Lemma 3-1 gives $\mathcal{S}$. Composing $\mathcal{S}$ with A gives $\mathcal{A}$, and composing B with $\mathcal{S}$ and $\mathcal{A}$ gives $\mathcal{B}$.

Now, if these 4 functions are related to the same potential, and are small as stated, then their relationships are integral equations which have unique solutions. So given that there is some (small) potential q, and that it has small scattering function $r^{0\pm} \in W^2(\mathbf{R}^{\pm})$, then its other scattering data must be the ones constructed by this theorem. $\square$

This immediately solves the short-time Cauchy problem for the Benjamin–Ono evolution. The evolution remains bounded for a while because the function $r^{0\pm}$ from which we can reconstruct q grows slowly under the corresponding linear evolution.

Corollary 4-5. *If q is a small, smooth complex-valued potential, then there is some $\epsilon > 0$ such that for time less than ϵ there is a solution to the Benjamin–Ono initial-value problem with initial data q.*

Proof. From the integral equations we see that $r^{0\pm}$ is approximately the derivative of $s^{0\pm}$, which will be shown in the next section to evolve by $s^{0\pm}(\zeta,t) = s^{0\pm}(\zeta,0)\exp(\pm it\zeta^2)$. Hence $|r^{0\pm}(\zeta,t)| < c|\zeta t r^{0\pm}(\zeta,0)|$. If q is small with three derivatives, then its associated $r^{0\pm}$ decays like $1/\zeta$ as $\zeta \to \infty$. Thus for $0 \le t < \epsilon$, $r^{0\pm}(\cdot,t)$ remains small enough to insure the existence of $q(\cdot,t)$. □

Other flows in the manifold of scattering data pull back under the scattering transform, in general to nonlinear flows of the associated potentials. There are four tangent maps calculable from the pullbacks from q to the scattering data:

(4.16)
$$\begin{aligned}\dot\alpha^{0\pm}(z,t) &= \int_{\mathbf{R}} \big(q(x,t)m^{0\pm}(x,z,t)\big)^{\cdot}\, dx = \int_{\mathbf{R}} \left(\dot q m^{0\pm} + q\frac{d}{dt}(1-\mathbf{G}_z^{0\pm}\mathbf{q})^{-1}1\right) dx \\ &= \int_{\mathbf{R}} \left(\dot q m^{0\pm} + q(1-\mathbf{G}_z^{0\pm}\mathbf{q})\mathbf{G}_z^{0\pm}\dot{\mathbf{q}}m^{0\pm}\right) dx \\ &= \int_{\mathbf{R}} \left(1-\mathbf{q}\mathbf{G}_z^{0\pm}\right)^{-1}(\dot{\mathbf{q}}m^{0\pm})\, dx = \int_{\mathbf{R}} \left[(1-{}^t\mathbf{G}_z^{0\pm}\mathbf{q})^{-1}1\right] q m^{0\pm}\, dx \\ &= \int_{\mathbf{R}} m^{0\mp}(x,-z,t)\dot q(x,t) m^{0\pm}(x,z,t)\, dx.\end{aligned}$$

Virtually identical calculations give the following:

$$\dot\beta^{0\pm}(z,t) = \int_{\mathbf{R}} m^{0\mp}(x,-z,t)\dot q(x,t)p^{0\pm}(x,z,t)\, dx, \tag{4.17}$$

$$\dot s^{0\pm}(\zeta,t) = \int_{\mathbf{R}} n^{0\mp}(x,-\zeta,t)\dot q(x,t)m^{0\pm}(x,\zeta+,t)\, dx, \tag{4.18}$$

$$\dot r^{0\pm}(\zeta,t) = \int_{\mathbf{R}} n^{0\mp}(x,-\zeta,t)\dot q(x,t)p^{0\pm}(x,\zeta+,t)\, dx. \tag{4.19}$$

These integrals converge absolutely if q is small, with enough decay to guarantee the existence of $\alpha^{0\pm}$, etc. The tangent maps may be used to find other nonlinear evolution equations linearizable by the same scattering—inverse scattering transform.

5. Extra Relations and the Selfadjoint Case

The results of previous sections apply to small but complex-valued potentials q. If q is real valued, then the operators V_q and $\mathbf{G}_z^0\mathbf{q}$ are selfadjoint. Given this extra hypothesis, it is possible to use the improved spectral quantities $m^{0\pm}, \alpha^{0\pm}$, etc., to estimate the original spectral quantities for the unimproved equations. In the process, the range of validity of formal conservation laws becomes evident, and the singular spectrum of the original operator may be described in detail.

Green's function $G_z^{\pm}$ for the original integral equation (1.1) satisfies the following symmetry:

$$\overline{G_z^{\pm}(x)} = \frac{1}{2\pi}\int_0^{\pm\infty} \frac{e^{-ix\xi}}{\xi - \bar z}\, d\xi = \frac{1}{2\pi}\int_0^{\mp\infty} \frac{e^{ix\xi}}{\xi + \bar z}\, d\xi = G^{\mp}_{-\bar z}(x). \tag{5.1}$$

With Eq.(2.5), this gives

$$\overline{\mathbf{G}_z^{\pm}} = \mathbf{G}_{-\bar{z}}^{\mp} = {}^t\mathbf{G}_{\bar{z}}^{\pm}. \tag{5.2}$$

If χ is real-valued in addition to its other virtues, then $l^{\pm}$ behaves like $G_z^{\pm}$:

$$\overline{l^{\pm}(z)} = \int_0^{\pm\infty} \overline{\left(\frac{\chi(\xi)}{\xi - z}\right)}\, d\xi = \int_0^{\pm\infty} \frac{\chi(\xi)}{\xi - \bar{z}}\, d\xi = l^{\mp}(-\bar{z}) = l^{\pm}(\bar{z}). \tag{5.3}$$

Thus $\mathbf{G}_z^0$ behaves like $\mathbf{G}_z$. Using Eq.(2.6) gives:

$$\overline{G_z^{0\pm}(x)} = G_{-\bar{z}}^{0\mp}(x) \implies \overline{\mathbf{G}_z^{0\pm}} = \mathbf{G}_{-\bar{z}}^{0\mp} = {}^t\mathbf{G}_{\bar{z}}^{0\pm}. \tag{5.4}$$

If $q = \bar{q}$, then the following relations hold among the eigenfunctions:

$$\begin{aligned} \overline{m^{0\pm}(x,z)} &= \overline{(I - \mathbf{G}_z^{0\pm}\mathbf{q})^{-1}}1 = (I - \mathbf{G}_{-\bar{z}}^{0\mp}\mathbf{q})^{-1}1 \\ &= m^{0\mp}(x, -\bar{z}), \end{aligned} \tag{5.5}$$

$$\begin{aligned} \overline{n^{0\pm}(x,\zeta)} &= \overline{(I - \mathbf{e}^*\mathbf{G}_{\zeta-}^{0\pm}\mathbf{e}\mathbf{q})^{-1}}1 = (I - \mathbf{e}\mathbf{G}_{(-\zeta)-}^{0\mp}\mathbf{e}^*\mathbf{q})^{-1}1 \\ &= n^{0\mp}(x, -\zeta), \end{aligned} \tag{5.6}$$

$$\begin{aligned} \overline{p^{0\pm}(x,z)} &= \left(I - \overline{\mathbf{G}_z^{0\pm}}\mathbf{q}\right)^{-1}\overline{(ix)} = \left(I - \mathbf{G}_{-\bar{z}}^{0\mp}\mathbf{q}\right)^{-1}(-ix) \\ &= -p^{0\mp}(x, -\bar{z}). \end{aligned} \tag{5.7}$$

From these follow relations between scattering data and their conjugates:

$$\begin{aligned} \overline{\alpha^{0\pm}(z)} &= \int_R q(x)\overline{m^{0\pm}(x,z)}\, dx = \int_{\mathbf{R}} q(x) m^{0\mp}(x, -\bar{z})\, dx \\ &= \alpha^{0\mp}(-\bar{z}) = \alpha^{0\pm}(\bar{z}), \end{aligned} \tag{5.8}$$

$$\overline{\beta^{0\pm}(z)} = \int_{\mathbf{R}} q(x)\overline{p^{0\pm}(x,z)}\, dx = -\int_{\mathbf{R}} q(x) p^{0\mp}(x, -\bar{z})\, dx = -\beta^{0\mp}(-\bar{z}), \tag{5.9}$$

$$\begin{aligned} \overline{s^{0\pm}(\zeta)} &= \int_R \overline{e^{-ix\zeta} q(x) m^{0\pm}(x, \zeta+)}\, dx = \int_R e^{ix\zeta} q(x) \overline{m^{0\pm}(x, \zeta+)}\, dx \\ &= \int_{\mathbf{R}} e^{ix\zeta} q(x) m^{0\mp}(x, -(\overline{\zeta+}))\, dx = \int_{\mathbf{R}} e^{ix\zeta} q(x) m^{0\mp}(x, (-\zeta)+)\, dx \\ &= s^{0\mp}(-\zeta), \end{aligned} \tag{5.10}$$

$$\begin{aligned} \overline{r^{0\pm}(\zeta)} &= \int_{\mathbf{R}} \overline{e^{-ix\zeta} q(x) p^{0\pm}(x, \zeta+)}\, dx = \int_{\mathbf{R}} e^{ix\zeta} q(x) p^{0\mp}(x, -\overline{(\zeta+)})\, dx \\ &= -\int_{\mathbf{R}} e^{ix\zeta} q(x) p^{0\mp}(x, (-\zeta)+)\, dx = -r^{0\mp}(-\zeta). \end{aligned} \tag{5.11}$$

Notice in Eqs. (5.11) and (5.10) that a single half of s^0 or r^0 determines the other half. For example, it is enough to know $s^{0+}(\zeta), \zeta > 0$ in order to know $s^{0-}(\zeta), \zeta < 0$. This is a property shared with the Fourier transform of a real-valued function.

Notice too that $\alpha^{0\pm}$ is symmetric with respect to complex conjugation. The same is true for the fixed functions $l^\pm$ and $k^\pm$.

In the selfadjoint case, even the "bad" original scattering data is well behaved. Consider the relations between original and improved data in which the denominator is $1 - l^\pm\alpha^{0\pm}$. We can show that whenever it vanishes, the numerator vanishes fast enough to cancel out the singularity. This fact is a consequence of Eq.(4.3). If the potential q is real-valued, then in that equation the right-hand side is bounded. Thus, the division by $1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)$ does not introduce any singularities into the left-hand side. In fact, there is an estimate:

$$\frac{|l^\pm(\zeta-)||s^{0\pm}(\zeta)|^2}{|1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)|} \le |1 - i\ \mathrm{sgn}\,\zeta\,\chi(\zeta)\alpha^{0\pm}(\zeta+)| + 1. \tag{5.12}$$

Since the denominators are the same, the following estimate is equivalent:

$$\frac{|l^\pm(\zeta-)||s^{0\pm}(\zeta)|^2}{|1 - l^\pm(\zeta-)\alpha^{0\pm}(\zeta-)|} \le |1 - i\ \mathrm{sgn}\,\zeta\,\chi(\zeta)\alpha^{0\pm}(\zeta+)| + 1. \tag{5.13}$$

Recalling the circumstances under which $s^{0\pm}$ exists gives a basic lemma:

Lemma 5-1. *If $w^n q$ is small in $L^1(\mathbf{R})$ for some $n > 0$, and q is real-valued, then $s^{0\pm}$ vanishes at each $\zeta \in \mathbf{R}$ for which $1 - l^\pm(\zeta-)\alpha^{0\pm}(\zeta-) = 0$ or $1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+) = 0$.*

Proof. $s^{0\pm}$ and $\alpha^{0\pm}$ exist in W for q as given, by Proposition 1-5, Proposition 1-6, and Proposition 1-7. The boundedness of Eqs.(5.12) and (5.13) gives the result. □

Define the sets $Z^\pm = \{\zeta \in \mathbf{R} : 1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+) = 1 - l^\pm(\zeta-)\alpha^{0\pm}(\zeta-) = 0\}$. By the lemma, $s^{0\pm}$ vanishes on $Z^\pm$. In fact, if q has enough decay that $r^{0\pm}$ exists, then $s^{0\pm}$ vanishes to high order on $Z^\pm$. This may be proved by a series of lemmas, each interesting in their own right:

Lemma 5-2. $0 \notin Z^\pm$.

Proof. If $0 \in Z^\pm$, then $\lim_{\zeta\to 0} l^\pm(\zeta+)\alpha^{0\pm}(\zeta+) = 1$. This requires that $\alpha^{0\pm}(\zeta+) \to 0$ as $\zeta \to 0$. But since $\alpha^{0\pm}$ is Hölder continuous, there is some $\delta > 0$ such that $|\alpha^{0\pm}(\zeta+)| \le \|\alpha^{0\pm}\|_{\mathrm{Lip}_\delta}|\zeta|^\delta$. And so $l^\pm(\zeta+) \sim -\log\zeta$ yields a contradiction: $\lim_{\zeta\to 0} l^\pm(\zeta+)\alpha^{0\pm}(\zeta+) = 0$. □

Recall the notation $D_\zeta = \partial/\partial\zeta$:

Lemma 5-3. *Fix $1 \le p \le \infty$ and set $n > (p-1)/p$. If $w^{n+1}q$ is small in $L^p(\mathbf{R})$, and q is real-valued, then $D_\zeta s^{0\pm}$ exists and vanishes at each $\zeta_0 \in Z^\pm$.*

Proof. Rearranging Eq.(4.6) gives:

$$\begin{aligned} D_\zeta s^{0\pm}(\zeta) = &- \left(1 - l^\mp((-\zeta)-)\alpha^{0\mp}((-\zeta)-)\right) r^{0\pm}(\zeta) \\ &- \left[k^\mp((-\zeta)-)\alpha^{0\mp}((-\zeta)-) + l^\mp((-\zeta)-)\beta^{0\mp}((-\zeta)-)\right] s^{0\pm}(\zeta). \end{aligned}$$

Using selfadjointness via Eqs.(5.3) and (5.8) shows that $1-l^\mp((-\zeta)-)\alpha^{0\mp}((-\zeta)-) = 1-l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)$. When q has the hypothesized decay, $r^{0\pm}$ is bounded and the first term on the right-hand side vanishes on $Z^\pm$. The second term is bounded on $Z^\pm$, and the result follows from Lemma 5-1. □

Lemma 5-4. *$D_\zeta\left(1 - l^{\pm}\alpha^{0\pm}\right)(\zeta+) = -1/(2\pi\zeta l^{\pm}(\zeta+)) \neq 0$ if $\zeta \in Z^{\pm}$.*

Proof. First find the derivative of $\alpha^{0\pm}(\zeta+)$ with respect to ζ:

$$\begin{aligned}
D_\zeta\alpha^{0\pm}(\zeta+) &= D_\zeta \int_{\mathbf{R}} q(x) m^{0\pm}(x,\zeta+)\,dx \\
&= \int_{\mathbf{R}} q(x)[D_\zeta - ix]m^{0\pm}(x,\zeta+)\,dx + \int_{\mathbf{R}} ix\, q(x) m^{0\pm}(x,\zeta+)\,dx \\
&= \int_{\mathbf{R}} q(x)[D_\zeta - ix]\left(I - \mathbf{G}^{0\pm}_{\zeta+}\mathbf{q}\right)^{-1} 1\,dx + \beta^{0\mp}((-\zeta)-) \\
&= \int_{\mathbf{R}} q(x)\left[D_\zeta - ix, \left(I - \mathbf{G}^{0\pm}_{\zeta+}\mathbf{q}\right)^{-1}\right] 1\,dx \\
&\qquad - \int_{\mathbf{R}} q(x)\left(I - \mathbf{G}^{0\pm}_{\zeta+}\mathbf{q}\right)^{-1}(ix)\,dx + \beta^{0\mp}((-\zeta)-) \\
&= \int_{\mathbf{R}} q(x)\left(I - \mathbf{G}^{0\pm}_{\zeta+}\mathbf{q}\right)^{-1}\left[D_\zeta - ix, \mathbf{G}^{0\pm}_{\zeta+}\right]\mathbf{q} m^{0\pm}(x,\zeta+)\,dx \\
&\qquad - \beta^{0\pm}(\zeta+) + \beta^{0\mp}((-\zeta)-).
\end{aligned}$$

Now $\left[D_\zeta - ix, \mathbf{G}^{0\pm}_{\zeta+}\right] = \left[\mathbf{e}D_\zeta\mathbf{e}^*, \mathbf{G}^{0\pm}_{\zeta+}\right] = \mathbf{e}\left[D_\zeta, \mathbf{e}^*\mathbf{G}^{0\pm}_{\zeta+}\mathbf{e}\right]\mathbf{e}^*$, and this last was calculated in Eq.(2.25). Substituting that expression into the integral above yields:

$$\begin{aligned}
D_\zeta\alpha^{0\pm}(\zeta+) = &\int_{\mathbf{R}} q(x)\left(I - \mathbf{G}^{0\pm}_{\zeta+}\mathbf{q}\right)^{-1}\left(k^{\pm}(\zeta+)\int_{\mathbf{R}} \mathbf{q} m^{0\pm}(y,\zeta+)\,dy\right)dx \\
&+ \int_{\mathbf{R}} q(x)\left(I - \mathbf{G}^{0\pm}_{\zeta+}\mathbf{q}\right)^{-1}\left(ix\, l^{\pm}(\zeta+)\int_{\mathbf{R}} q(y) m^{0\pm}(y,\zeta+)\,dy\right)dx \\
&- \int_{\mathbf{R}} q(x)\left(I - \mathbf{G}^{0\pm}_{\zeta+}\mathbf{q}\right)^{-1}\left(l^{\pm}(\zeta+)\int_{\mathbf{R}} iy\, q(y) m^{0\pm}(y,\zeta+)\,dy\right)dx \\
&- \beta^{0\pm}(\zeta+) + \beta^{0\mp}((-\zeta)-)
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbf{R}} q(x)\left(I - \mathbf{G}^{0\pm}_{\zeta+}\mathbf{q}\right)^{-1}\left(k^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)\right)dx \\
&\qquad + \int_{\mathbf{R}} q(x)\left(I - \mathbf{G}^{0\pm}_{\zeta+}\mathbf{q}\right)^{-1} l^{\pm}(\zeta+)\left[ix\,\alpha^{0\pm}(\zeta+) - \beta^{0\mp}((-\zeta)-)\right]dx \\
&\qquad - \beta^{0\pm}(\zeta+) + \beta^{0\mp}((-\zeta)-) \\
&= k^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)^2 + l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)\left[\beta^{0\pm}(\zeta+) - \beta^{0\mp}((-\zeta)-)\right] \\
&\qquad - \beta^{0\pm}(\zeta+) + \beta^{0\mp}((-\zeta)-).
\end{aligned}$$

Finally, rearranging the terms a bit yields

$$D_\zeta\alpha^{0\pm}(\zeta+) = k^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)^2 - \left(1 - l^{\pm}(\zeta+)\alpha^{0\pm}(\zeta+)\right)\left[\beta^{0\pm}(\zeta+) - \beta^{0\mp}((-\zeta)-)\right]. \tag{5.14}$$

In particular, if q has enough decay that $p^{0\pm}$ and thus $\beta^{0\pm}$ exist, then the right-hand side of this equation simplifies at any root ζ_0 of $1 - l^{\pm}\alpha^{0\pm}$ to

$$D_\zeta\alpha^{0\pm}(\zeta_0+) = k^{\pm}(\zeta_0+)\alpha^{0\pm}(\zeta_0+)^2, \qquad \text{if } \zeta_0 \in Z^{\pm}.$$

Recall from Eq.(2.23) that $D_\zeta l^\pm = -k^\pm + 1/2\pi\zeta$, and that if $\zeta_0 \in Z^\pm$ then $\alpha^{0\pm}(\zeta_0+) = 1/l^\pm(\zeta_0+)$. This completes the calculation:

(5.15)
$$\begin{aligned} D_\zeta(1 - l^\pm\alpha^{0\pm})(\zeta_0+) &= -D_\zeta l^\pm(\zeta_0+)\alpha^{0\pm}(\zeta_0+) - l^\pm(\zeta_0+)D_\zeta\alpha^{0\pm}(\zeta_0+) \\ &= \left[k^\pm(\zeta_0+) - \frac{1}{2\pi\zeta_0} - k^\pm(\zeta_0+)l^\pm(\zeta_0+)\alpha^{0\pm}(\zeta_0+)\right]\alpha^{0\pm}(\zeta_0+) \\ &= \frac{-1}{2\pi\zeta_0 l^\pm(\zeta_0+)}. \quad \Box \end{aligned}$$

One consequence of the last two lemmas is the boundedness of "bad" scattering data:

Lemma 5-5. *Suppose that q is real and that $w^{n+1}q \in L^1$ for some $n > 0$. Then the ratio*

$$\frac{s^{0\pm}(\zeta)}{[1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)]^2} \tag{5.16}$$

is bounded and continuous on $\mathbf{R}$.

Proof. By Lemmas 5-3 and 5-4, both numerator and denominator are differentiable. Both vanish at each point of $Z^\pm$, the complete set of roots of the denominator. Hence L'Hôpital's rule may be used at each $\zeta_0 \in Z^\pm$. Indeed,

$$\lim_{\zeta\to\zeta_0} \frac{s^{0\pm}(\zeta)}{1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)} = \lim_{\zeta\to\zeta_0} \frac{D_\zeta s^{0\pm}(\zeta)}{D_\zeta[1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)]} = 0, \tag{5.17}$$

because the denominator does not vanish while the numerator does. But then,

$$\begin{aligned} \lim_{\zeta\to\zeta_0} \frac{s^{0\pm}(\zeta)}{[1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)]^2} &= \lim_{\zeta\to\zeta_0} \frac{D_\zeta s^{0\pm}(\zeta)}{D_\zeta[1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)]^2} \\ &= \lim_{\zeta\to\zeta_0} \frac{-\left(1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)\right) r^{0\pm}(\zeta)}{2[1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)]D_\zeta[1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)]} \\ &\quad - \lim_{\zeta\to\zeta_0} \frac{\left[k^\pm(\zeta+)\alpha^{0\pm}(\zeta+) + l^\pm(\zeta+)\beta^{0\mp}((-\zeta)-)\right] s^{0\pm}(\zeta)}{2[1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)]D_\zeta[1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+)]} \\ &= -\frac{r^{0\pm}(\zeta_0)}{2/(-2\pi\zeta_0 l^\pm(\zeta_0))} - 0 = \pi\zeta_0 l^\pm(\zeta_0+)r^{0\pm}(\zeta_0). \end{aligned}$$

This quantity is bounded for all $\zeta_0 \in \mathbf{R}$, because $\zeta \to 0$ faster than $l^\pm(\zeta+) \to \infty$, and $l^\pm(\zeta+) \to 0$ faster than $\zeta \to \infty$. Furthermore, its size is controlled by $cr^{0\pm}(\zeta_0)$, where $c = \sup_\zeta |\pi\zeta l(\zeta)| \simeq \frac{\pi}{e}$ depends only upon the renormalization, i.e., on χ.

Off $Z^\pm$, the ratio is continuous, hence locally bounded. But since the numerator is bounded and $1 - l^\pm(\zeta+)\alpha^{0\pm}(\zeta+) \to 1$ as $\zeta \to \infty$, in fact the ratio is uniformly bounded for all $\zeta \in \mathbf{R}$. $\Box$

An immediate corollary of this lemma is that the conservation law in Anderson and Taflin [AT] makes sense for all q of sufficient decay, in fact for $q(x) = O(x^{-2-\epsilon})$ for any $\epsilon > 0$. It states that whenever the integral makes sense, we have

$$\int_{\mathbf{R}} q(x)\,dx = -\int_{-\infty}^{\infty} \frac{|s(\zeta)|^2}{|\zeta|}\,d\zeta + \sum_{Z^+\cup Z^-} 2\pi, \tag{5.18}$$

But $|s(\zeta)|^2/|\zeta|$ is integrable in the selfadjoint case by the estimates

$$\frac{|s(\zeta)|^2}{|\zeta|} \leq \begin{cases} c_0/|\zeta|(\log|\zeta|)^2, & \text{as } \zeta \to 0, \\ c_\infty/|\zeta|^2, & \text{as } \zeta \to \pm\infty, \\ c_1 s^0(\zeta), & \text{uniformly in any compact set excluding } \{0, \pm\infty\}. \end{cases}$$

For the constant c_1, we can use the bound obtained in Lemma 5-5.

Combining the representation of q given by Eq. (3.30) with Lemma 5-5 also gives an alternative proof of the existence of the inverse scattering transform in the selfadjoint case:

Theorem 5-6. *Suppose that q is real and that $w^{n+1}q$ is a small function in L^1 for some $n > 0$. Suppose that $w^n q'$ and $w^n q''$ are also small in L^1. Then the eigenfunctions $n^{0\pm}$ and the scattering data $s^{0\pm}$ associated to q exist, and q may be represented by the integral in Eq.* (3.30)*, which converges absolutely.*

Proof. That $n^{0\pm}$ and $s^{0\pm}$ exist and are bounded is a consequence of Proposition 1-6. The differentiability of q guarantees by Proposition 2-2 that $s^{0\pm}$ decays like $1/\zeta^2$ as $|\zeta| \to \infty$. These estimates, together with the continuity of the integrand implied by Lemma 5-5, imply the result. □

Corollary 5-7. *If the initial data q satisfies the hypotheses of Theorem 5-6, then there exists a solution for all time to the corresponding initial value problem for the Benjamin–Ono equation.*

Proof. In the selfadjoint case, $q(x,t)$ for all $t > 0$ may be reconstructed from an absolutely convergent integral involving only $s^{0\pm}$. But the evolution of $s^{0\pm}$ corresponding to the BO evolution is the same as the evolution of $s^{\pm}$. Now, the following equations are consequences of the formulas in Chapter 2:

$$\dot{\alpha}^{0\pm}(z,t) = \left(\frac{\alpha(x,t)}{1 + l^{\pm}(z)\alpha(z,t)}\right)^{\cdot}, \tag{5.19}$$

$$\dot{s}^{0\pm}(\zeta,t) = \left(\frac{s^{\pm}(\zeta,t)}{1 + l^{\pm}(\zeta+)\alpha^{\pm}(\zeta+,t)}\right)^{\cdot}, \tag{5.20}$$

Obviously, $\dot{l}^{\pm} = 0$. Thus, if attention is restricted only to those evolutions for which $\dot{\alpha}^{\pm} = 0$, the above equations simplify:

$$\dot{\alpha}^{0\pm}(z,t) = 0, \tag{5.21}$$

$$\dot{s}^{0\pm}(\zeta,t) = \frac{\dot{s}^{\pm}(\zeta,t)}{1 + l^{\pm}(\zeta+)\alpha^{\pm}(\zeta+,t)}, \tag{5.22}$$

The Benjamin–Ono evolution corresponds to $s^{\pm}(\zeta,t) = s^{\pm}(\zeta,0)\exp(\pm it\zeta^2)$. Thus $s^{0\pm}$ remains small for all time $t > 0$, and the integral representing $q(x,t)$ remains absolutely convergent. □

While other investigators have shown global existence for the BO Cauchy problem, the above proof has the advantage of generalizing to complex-valued potentials, and showing exactly how selfadjointness guarantees long-time boundedness.

6. Bound States of the Spectral Problem, and Non-analyticity of the Scattering Transform.

Previous analyses of the scattering and inverse scattering transforms for the Benjamin–Ono equation have relied on the spectral theory of selfadjoint operators to study the associated eigenvalue problem. While the selfadjoint spectral problem that arises when q is real has better analytic properties than the general (complex) case, it is the complex potentials that determine the behavior of the direct and inverse series. In this section we present two results derived from studying such potentials.

First, we prove that even for complex potentials, the eigenvalues of the BO spectral problem cannot accumulate at the origin. This is an application of analyticity of the "improved" problem, and has not been shown by methods that rely on selfadjointness.

Second, we prove that the power series for the direct problem is not analytic. This is not because of the presence of bound states, however. Instead, it fails to be analytic because for generic (complex) potentials q of arbitrarily small size, there are nearby potentials for which s, α, and so on are arbitrarily large. Thus we are forced to use the "improved" scattering transform to obtain absolutely convergent series. In fact, the nonanalyticity of the direct problem may be proved from the representation of the "bad" scattering data in terms of the "improved" data. This is done at the end of this section.

We begin by showing that there is a small neighborhood of 0 in $\mathbf{C}$ in which there are no embedded eigenvalues.

Theorem 6-1. *If q has sufficient decay as $x \to \pm\infty$, then the Benjamin-Ono scattering problem has no bound states in a neighborhood of the origin.*

Proof. The result follows from two lemmas.

Lemma 6-2. *Suppose that $w^n q$ is a small function in $L^1(\mathbf{R})$ for some $n > 0$. If z is bound state eigenvalue for the Benjamin-Ono spectral problem, then $1 - l^{\pm}(z)\alpha^{0\pm}(z) = 0$.*

Proof. Suppose that $\psi(\cdot, z)$ is a bound state, namely a nonzero solution in $L^2(\mathbf{R})$, of the eigenvalue problem $-id\psi/dx - V_q\psi = z\psi$. Then ψ splits into two halves, $\psi = P^+\psi + P^-\psi$, where P^+ and P^- are the orthogonal projections of $L^2(\mathbf{R})$ onto H^2 and $\overline{H^2}$. Each of these halves belongs in turn to L^2, so that there exist bound states at z for at least one of the two eigenvalue problems below:

$$\frac{1}{i}\frac{d}{dx}\psi^+ - P^+\mathbf{q}\psi^+ = z\psi^+, \tag{6.1+}$$

$$\frac{1}{i}\frac{d}{dx}\psi^- + P^-\mathbf{q}\psi^- = z\psi^-. \tag{6.1−}$$

In particular, these $\psi^{\pm}$ must satisfy the integral equation below:

$$\psi^{\pm}(x,z) = \mathbf{G}_z^{\pm}\mathbf{q}\psi^{\pm}(x,z) = \mathbf{G}_z^{0\pm}\mathbf{q}\psi^{\pm}(x,z) + l^{\pm}(z)\int_{\mathbf{R}} q(y)\psi^{\pm}(y,z)\,dy, \tag{6.2}$$

$$\implies \psi^{\pm}(x,z) = (I - \mathbf{G}_z^{0\pm}\mathbf{q})^{-1}\left(l^{\pm}(z)\int_{\mathbf{R}} q(y)\psi^{\pm}(y,z)\,dy\right)$$

$$= m^{0\pm}(x,z)l^{\pm}(z)\int_{\mathbf{R}} q(y)\psi^{\pm}(y,z)\,dy, \tag{6.3}$$

for $m^{0\pm}(x,z)$ defined by Proposition 1-5. In particular, the asymptotic behavior of $\psi^{\pm}(x,z)$ as $|x| \to \infty$ is determined by that of $m^{0\pm}$. Now $l^{\pm}(z)\int_{\mathbf{R}} q(y)\psi^{\pm}(y,z)\,dy \neq 0$, for in that case $\psi^{\pm}(\cdot,z)$ vanishes identically, so that

$$0 \neq \psi^{\pm}(\cdot,z) \in L^2(\mathbf{R}) \implies m^{0\pm}(\cdot,z) \in L^2(\mathbf{R}). \tag{6.4}$$

But then, both sides of Eq.(6.3) may be multiplied by $q(x)$ and integrated with respect to x. This gives

$$\int_{\mathbf{R}} q(x)\psi^{\pm}(x,z) = \alpha^{0\pm}(z)l^{\pm}(z)\int_{\mathbf{R}} q(y)\psi^{\pm}(y,z)\,dy, \tag{6.5}$$

$$\implies \left(1 - l^{\pm}(z)\alpha^{0\pm}(z)\right)\int_{\mathbf{R}} q(y)\psi^{\pm}(y,z)\,dy = 0.$$

Since the integral does not vanish, $1 - l^{\pm}(z)\alpha^{0\pm}(z)$ must. □

But this quantity cannot vanish too close to $z = 0$:

Lemma 6-3. *Suppose that $w^n q$ is a small function in $L^1(\mathbf{R})$ for some $n > 0$. Then there is some neighborhood of $0 \in \mathbf{C}$ containing no zeroes of the expression $1 - l^{\pm}(z)\alpha^{0\pm}(z)$.*

Proof. The $\pm$ cases are identical. With q as given, the function $\alpha^{0\pm}$ exists and is Hölder continuous in z. There are two possibilities to consider. Either $\alpha^{0\pm}(0) = 0$, or $\alpha^{0\pm}(0) \neq 0$. In the first case, $l^{\pm}(z)\alpha^{0\pm}(z) \to 0$ as $z \to 0$. In the second case, $|l^{\pm}(z)\alpha^{0\pm}(z)| \to \infty$ as $z \to 0$. In either case there is some open ball around $0 \in \mathbf{C}$ for which $|1 - l^{\pm}(z)\alpha^{0\pm}(z)| > 1/2$. □

Combining these lemmas shows that there is some neighborhood of $0 \in \mathbf{C}$ containing no bound states of the eigenvalue problem (6.1±), completing the proof of the theorem. □

Note that for any $\epsilon > 0$, $q = O((1+|x|)^{-1-\epsilon})$ is sufficient decay for the absence of bound states near 0. Note too that this result holds for small complex-valued q, as well as for real-valued q. In particular, the absence of bound states near 0 for any given potential is a property that that does not depend upon selfadjointness.

Even though for any fixed small q the determinant $1 - l^{\pm}(z)\alpha^{0\pm}(z)$ never vanishes near 0, there are other small q's for which it gets arbitrarily small arbitrarily close to 0. Such behavior destroys the analyticity of the scattering map. In fact, the maps $q \mapsto s$, $q \mapsto m$, and so on are not analytic without renormalization. We will prove this only for the datum s, which in other papers is the primary scattering function, but the other cases are similar. In particular, we will show the following:

Theorem 6-4. *Given any $\epsilon > 0$, for any sufficiently small $z \in \mathbf{R}$, there is a potential q smaller than ϵ for which the corresponding s is larger that $|\log z|$ at z. The chosen q is generic in the sense that its associated α^0 merely has to be nonvanishing at 0.*

Proof. From Eq.(4.3), the boundedness of $s(z)$ depends upon that of

$$\frac{1 - l(z+)\alpha^0(z+)}{1 - l(z-)\alpha^0(z-)}.$$

But if q is any small function for which $\alpha^0(0) = \int_{\mathbf{R}} q(x)m^0(x,0)\,dx \neq 0$, then the analyticity of the map $q \mapsto \alpha^0$ plus the smoothness of α^0 in z assures that the maps $w \mapsto \alpha^0(z\pm)$ are open in a neighborhood $\omega \in \mathbf{C}$, $|\omega| \leq 1$. Furthermore, then $\alpha^0(z-)$ will differ from $\alpha^0(z+)$ by $O(|\alpha^0(z)|^2 + |s^0(z)|^2)$, which is negligible compared to $|\alpha^0(z)|$. So write $\alpha^0(z+) = \alpha^0(z-) = \xi + i\eta$.

Now choose $z \in \mathbf{R}$ so small that $l(z+) = i + \log|z|$, $l(z-) = -i + \log|z|$, and so that $1/l(z\pm)$ is in the range of $\omega \mapsto \alpha^0(z\pm)$. This is possible since $\alpha^0(0) \neq 0$, and $|\log z| \to \infty$ as $z \to 0$. Choosing the appropriate ω, set $\xi = 1/\log|z|$ and $\eta = -\xi/\log|z|$. This gives the worst possible behavior of the ratio above, as estimated here:

$$\frac{1 - l(z+)\alpha^0(z+)}{1 - l(z-)\alpha^0(z-)} \simeq \frac{1 - (i + \log|z|)(\xi + i\eta)}{1 - (-i + \log|z|)(\xi + i\eta)} = \frac{1 - \xi\log|z| + \eta - i(\xi - \eta\log|z|)}{1 - \xi\log|z| - \eta - i(-\xi - \eta\log|z|)}$$
$$= \frac{\eta - 2i\eta\log|z|}{-\eta} = 2i\log|z| - 1.$$

As $z \to 0$, this blows up in absolute value. □

The genericity condition on q is satisfied even by almost all functions with infinitely many vanishing moments. This follows from the series for α^0:

$$\alpha^0(0) = \int_{\mathbf{R}} q(x)\,dx + \int_{\mathbf{R}} q(x)\mathbf{G}_z^0 q(x)\,dx + \ldots$$

No matter how many moments of q vanish, the product-convolution operators $q\mathbf{G}_z^{0\pm}$ will in general destroy the cancellation. The vanishing of $\alpha^0(0)$ requires imposing infinitely many nonlinear constraints on q. Hence the formal series scattering transform cannot be made analytic by restriction to a reasonable linear submanifold.

The formal inverse series contains even more drastic difficulties. It cannot be made analytic by restriction to a linear submanifold of scattering data, because by Eq.(2.13) and Proposition 2-3, the formal scattering data $s^{\pm}$ behaves exactly like $1/\log|\zeta|$ as $\zeta \to 0$. Linear operations immediately destroy this property. By expressing the formal data as rational functions of improved data, we have instead found a parametrization of these formal data.

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