# SMOOTH LOCALIZED ORTHONORMAL BASES

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Abstract. We describe an orthogonal decomposition of $L^2(\mathbf{R})$ which maps smooth functions to smooth periodic functions. It generalizes previous constructions by Malvar, Coifman and Meyer. The adjoint of the decomposition can be used to construct smooth orthonormal windowed exponential, wavelet and wavelet packet bases.

Orthogonal projections which map smooth functions to smooth compactly supported functions appeared in the work of Malvar [M] and Coifman and Meyer [CM]. In those papers the projections were used to build a smooth overlapping orthogonal basis on the line, composed of windowed sine (or cosine) functions. Many of these bases' properties were only briefly described in the short papers of Malvar, Coifman, and Meyer, but are developed in detail in [AWW]. We do not wish to overlook the original sources, but we will take advantage of some of the later paper's structure and notation. In this short note we sketch a construction of smoothness-preserving unitary maps onto periodic functions. We thereby show that arbitrary smooth periodic bases can be used as smooth "windowed" orthonormal bases on the line. We evade the Balian–Low obstruction by a modification of the definition of "window." The main ingredient in the recipe is a pair of orthogonal projections which can be factored into simple pieces. Let $r = r(t)$ be a function in the class $C^d(\mathbf{R})$ for some $0 \le d \le \infty$, satisfying the following conditions:

$$(1) \qquad |r(t)|^2 + |r(-t)|^2 = 1 \quad \text{for all } t \in \mathbf{R}; \qquad r(t) = \begin{cases} 0, & \text{if } t \le -1, \\ 1, & \text{if } t \ge 1; \end{cases}$$

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A general construction for such functions is given in [AWW]. We now define the *folding* operator $U = U(r)$ and its adjoint *unfolding* operator $U^* = U^*(r)$:

$$Uf(t) = \begin{cases} r(t)f(t) + r(-t)f(-t), & \text{if } t > 0, \\ \overline{r(-t)}f(t) - \overline{r(t)}f(-t), & \text{if } t < 0; \end{cases} \tag{2}$$

$$U^*f(t) = \begin{cases} \overline{r(t)}f(t) - r(-t)f(-t), & \text{if } t > 0, \\ r(-t)f(t) + \overline{r(t)}f(-t), & \text{if } t < 0. \end{cases} \tag{3}$$

We observe that $Uf(t) = f(t)$ and $U^*f(t) = f(t)$ if $t \geq 1$ or $t \leq -1$. Also, $U^*Uf(t) = UU^*f(t) = \left(|r(t)|^2 + |r(-t)|^2\right)f(t) = f(t)$ for all $t \neq 0$, so that $U$ and $U^*$ are unitary isomorphisms of $L^2(\mathbf{R})$. It does not matter how we define $Uf(0)$ or $U^*f(0)$ for functions $f \in L^2$; for smooth $f$ we may just as well define $Uf(0) \stackrel{\text{def}}{=} f(0)$, and for $f$ satisfying certain smoothness and boundary limit conditions we will show that there is a unique smooth extension of $U^*f$ across $t = 0$.

**Lemma 1.** *Suppose $r \in C^d(\mathbf{R})$ for $0 \leq d \leq \infty$. If $f \in C^d(\mathbf{R})$, then $Uf \in C^d(\mathbf{R} \setminus \{0\})$, the limits $[Uf]^{(n)}(0+)$ and $[Uf]^{(n)}(0-)$ exist for all $0 \leq n \leq d$, and we have:*

$$[Uf]^{(n)}(0+) = 0 \quad \textit{if } n \textit{ is odd}, \qquad [Uf]^{(n)}(0-) = 0 \quad \textit{if } n \textit{ is even}. \tag{4}$$

*Conversely, if $f \in C^d(\mathbf{R} \setminus \{0\})$ has limits $f^{(n)}(0+)$ and $f^{(n)}(0-)$ for all $0 \leq n \leq d$ which satisfy the equations*

$$f^{(n)}(0+) = 0 \quad \textit{if } n \textit{ is odd}, \qquad f^{(n)}(0-) = 0 \quad \textit{if } n \textit{ is even}, \tag{5}$$

*then $U^*f$ has a unique continuous extension (across $t = 0$) which belongs to $C^d(\mathbf{R})$.* □

This lemma shows that just a trivial boundary condition is needed to obtain smoothness. In particular, the 0 function satisfies the condition, and we shall use this fact to clarify the construction of the smooth orthogonal projections in [AWW]. For real numbers $\alpha$ and $\epsilon > 0$, define the translation and rescaling operators:

$$\begin{aligned} \tau_\alpha f(t) &= f(t+\alpha); & \tau_\alpha^* f(t) &= f(t-\alpha); \\ \delta_\epsilon f(t) &= \sqrt{\epsilon} f(\epsilon t); & \delta_\epsilon^* f(t) &= \frac{1}{\sqrt{\epsilon}} f(t/\epsilon). \end{aligned} \tag{6}$$

Then conjugation with $\delta_\epsilon$ and $\tau_\alpha$ dilates and translates the "range of influence" of the folding operators to an arbitrary interval $(\alpha-\epsilon,\ \alpha+\epsilon)$. We obtain a family of folding (respectively unfolding) operators indexed by $(r,\alpha,\epsilon)$:

$$U(r,\alpha,\epsilon)=\tau_\alpha^*\delta_\epsilon^*U(r)\delta_\epsilon\tau_\alpha;\qquad U^*(r,\alpha,\epsilon)=\tau_\alpha^*\delta_\epsilon^*U^*(r)\delta_\epsilon\tau_\alpha. \tag{7}$$

Where convenient we will write $U_0$ for $U(r_0,\alpha_0,\epsilon_0)$, and so on. We note that if $r_0$ and $r_1$ satisfy Eq.(1), and the intervals $(\alpha_0-\epsilon_0,\ \alpha_0+\epsilon_0)$ and $(\alpha_1-\epsilon_1,\ \alpha_1+\epsilon_1)$ are disjoint, then the operators $U_0$, $U_1$, $U_0^*$ and $U_1^*$ all commute. In this case we will say that $(r_0,\alpha_0,\epsilon_0)$ and $(r_1,\alpha_1,\epsilon_1)$ satisfy the *consistency conditions*. For such triplets, the projection $P_{(\alpha_0,\alpha_1)}$ of [AWW] factors as follows:

$$P_{(\alpha_0,\alpha_1)}=U_0^*U_1^*\chi_{(\alpha_0,\alpha_1)}U_1U_0, \tag{8}$$

where $\chi_I f(t)=f(t)$ if $t\in I$, and $\chi_I f(t)=0$ otherwise. The map $P_I$ replaces the traditional windowing of periodic bases. If we use a single $\epsilon$ and a single $r$, then the interval $I=(\alpha_0,\alpha_1)$ and its adjacent translate $J=(\alpha_0+|I|,\ \alpha_1+|I|)$ are compatible, i.e., $P_I+P_J=P_{I\cup J}$ and $P_IP_J=0$. The main consequence of this observation is that $P_I$ composed with $|I|$-periodization is a unitary isomorphism from $P_IL^2(\mathbf{R})$ to $L^2(I)$. We make this more precise by defining the $\lambda$-*periodization* $\Omega_\lambda f$ of a function $f=f(t)$ by the formula $\Omega_\lambda f(t)\stackrel{\text{def}}{=}\sum_{k\in\mathbf{Z}}f\left(t+k\lambda\right)=\sum_{k\in\mathbf{Z}}\tau_{k\lambda}f(t)$. If $f$ belongs to $L^2(\mathbf{R})$ and is compactly supported, then $\Omega_\lambda f$ belongs to $L^2_{loc}(\mathbf{R})$ and is periodic of period $\lambda$. If in addition $f$ belongs to $C^d(\mathbf{R})$, then $\Omega_\lambda f$ also belongs to $C^d(\mathbf{R})$.

We can now define a "periodized" version of the folding and unfolding operators. The periodization is hidden in the definition of the following operators:

$$W\left(r,(\alpha_0,\alpha_1),\epsilon\right)f(t)=\begin{cases} r(\frac{t-\alpha_0}{\epsilon})f(t)+r(\frac{\alpha_0-t}{\epsilon})f(\alpha_0+\alpha_1-t), & \text{if } \alpha_0<t\le\alpha_0+\epsilon,\\ \overline{r(\frac{\alpha_1-t}{\epsilon})}f(t)-\overline{r(\frac{t-\alpha_1}{\epsilon})}f(\alpha_0+\alpha_1-t), & \text{if } \alpha_1-\epsilon\le t<\alpha_1,\\ f(t), & \text{otherwise;}\end{cases} \tag{9}$$

$$W^*\left(r,(\alpha_0,\alpha_1),\epsilon\right)f(t)=\begin{cases} \overline{r(\frac{t-\alpha_0}{\epsilon})}f(t)-r(\frac{\alpha_0-t}{\epsilon})f(\alpha_0+\alpha_1-t), & \text{if } \alpha_0<t\le\alpha_0+\epsilon,\\ r(\frac{\alpha_1-t}{\epsilon})f(t)+\overline{r(\frac{t-\alpha_1}{\epsilon})}f(\alpha_0+\alpha_1-t), & \text{if } \alpha_1-\epsilon\le t<\alpha_1,\\ f(t), & \text{otherwise.}\end{cases} \tag{10}$$

For these to be well defined, we must assume that $(r, \alpha_0, \epsilon)$ and $(r, \alpha_1, \epsilon)$ satisfy the consistency conditions. Then using $\Omega$, we can write the relationship between $W$ and $U$. For all $t \in I = (\alpha_0, \alpha_1)$, we have:

$$\begin{aligned} W(r, I, \epsilon)\, f(t) &= U(r, \alpha_0, \epsilon) U(r, \alpha_1, \epsilon) \Omega_{|I|} \chi_I\, f(t); \\ W^*(r, I, \epsilon)\, f(t) &= U^*(r, \alpha_0, \epsilon) U^*(r, \alpha_1, \epsilon) \Omega_{|I|} \chi_I\, f(t). \end{aligned} \tag{11}$$

When there is no possibility of confusion, we will write $W_I$ for $W(r, I, \epsilon)$ where $I = (\alpha_0, \alpha_1)$, suppressing the $r$ and $\epsilon$. We observe that $W_I$ and $W_I^*$ are unitary isomorphisms of $L^2(\mathbf{R})$ (i.e., $W_I^* W_I = W_I W_I^* = Id$) because $|r(t)|^2 + |r(-t)|^2 = 1$ for all $t$. Also, if $t < \alpha_0$ or $t > \alpha_1$, then $W_I f(t) = f(t)$ and $W_I^* f(t) = f(t)$. Thus if $I$ and $J$ are disjoint intervals, the operators $W_I$, $W_I^*$, $W_J$, and $W_J^*$ all commute. We also note that if $f$ is smooth in the interval $I$, then $W_I f$ is also smooth there. Furthermore, $W_I f$ satisfies the same boundary conditions at $\alpha_0+$ and $\alpha_1-$ as $U_0 U_1 f$. If $f$ is periodic and satisfies Eq.(5) at $\alpha_0+$ and $\alpha_1-$, then $W_I^* f$ satisfies the equation:

$$\lim_{t \to \alpha_0+} \left[W_I^* f\right]^{(n)}(t) = \lim_{t \to \alpha_1-} \left[W_I^* f\right]^{(n)}(t), \qquad \text{for all } 0 \le n \le d. \tag{12}$$

Thus $W_I^* f$ has a continuous periodic extension in $C^d(\mathbf{R})$.

The first main point of this paper is that from arbitrary smooth *periodic* orthonormal bases we can construct smooth *compactly supported* orthonormal bases:

**Theorem 2.** *Suppose that $(r, \alpha_0, \epsilon), (r, \alpha_1, \epsilon)$ and $(r_0, \alpha_0, \epsilon_0), (r_1, \alpha_1, \epsilon_1)$ satisfy the consistency conditions, and write $I = (\alpha_0, \alpha_1)$ and $W_I = W(r, I, \epsilon)$. If $\{e_j\}_{j \in \mathbf{Z}}$ is a collection of $|I|$-periodic functions which form an orthonormal basis for $L^2(I)$ when restricted to $I$, then $E_0 = \{U_0^* U_1^* \chi_I W_I\, e_j\}_{j \in \mathbf{Z}}$ is an orthonormal basis of $P_I L^2(\mathbf{R})$. In addition, if $\{e_j\}_{j \in \mathbf{Z}} \subset C^d(\mathbf{R})$, then $E_0 \subset C_0^d(\mathbf{R})$.* $\square$

We can get an orthonormal basis for $L^2(\mathbf{R})$ out of this by decomposing $\mathbf{R}$ into adjacent compatible intervals. We write $\mathbf{R} = \bigcup_{k \in \mathbf{Z}} I_k \stackrel{\text{def}}{=} \bigcup_{k \in \mathbf{Z}} [a_k, a_{k+1})$ and define $U_k^* = U^*(r_k, \alpha_k, \epsilon_k)$ and $W_k = W(\tilde{r}_k, I_k, \tilde{\epsilon}_k)$.

**Corollary 3.** *Suppose that for each $k \in \mathbf{Z}$, the triplets $(r_k, \alpha_k, \epsilon_k), (r_{k+1}, \alpha_{k+1}, \epsilon_{k+1})$ and $(\tilde{r}_k, \alpha_k, \tilde{\epsilon}_k), (\tilde{r}_k, \alpha_{k+1}, \tilde{\epsilon}_k)$ satisfy the consistency conditions, and the family of $|I_k|$-periodic functions $\{e_{k,j} : j \in \mathbf{Z}\}$ forms an orthonormal basis of $L^2(I_k)$ when restricted to $I_k$; Then* *the collection* $E = \{U_k^* U_{k+1}^* \chi_{I_k} W_k\, e_{k,j} \ :\ j,k \in \mathbf{Z}\}$ *is an orthonormal basis for* $L^2(\mathbf{R})$ *consisting of functions of compact support. If in addition all the functions* $e_{k,j}$ *and* $r_k$, $k,j \in \mathbf{Z}$, *belong to* $C^d(\mathbf{R})$ *for some* $0 \le d \le \infty$, *then the functions in* $E$ *belong to* $C_0^d(\mathbf{R})$.

*Proof.* Adjacent intervals $I_k, I_{k+1}$ are compatible, so $L^2(\mathbf{R}) = \bigoplus_{k\in\mathbf{Z}} P_{I_k} L^2(\mathbf{R})$. Each of the spaces $P_I L^2(\mathbf{R})$ has an orthonormal basis $E_k = \{U_k^* U_{k+1}^* \chi_{I_k} W_k\, e_{k,j} : j \in \mathbf{Z}\}$. Putting these bases together into $E = \bigcup_{k\in\mathbf{Z}} E_k$ yields the result. □

The second main result is just the adjoint of Theorem2. There are better-tested computer programs for expanding a periodic function in a periodic basis, so the adjoint result is more useful in practice:

**Theorem 4.** *Suppose that the triplets* $(r_0, \alpha_0, \epsilon_0), (r_1, \alpha_1, \epsilon_1)$ *and* $(r, \alpha_0, \epsilon), (r, \alpha_1, \epsilon)$ *satisfy the consistency conditions. Write* $I = (\alpha_0, \alpha_1)$, $U_0 = U(r_0, \alpha_0, \epsilon_0)$, $U_1 = U(r_1, \alpha_1, \epsilon_1)$, *and* $W_I^* = W^*(r, I, \epsilon)$. *If* $f$ *belongs to* $C^d(\mathbf{R})$, *then* $T_I f \stackrel{\text{def}}{=} W_I^* \chi_I U_0 U_1\, f$ *has an* $I$*-periodic extension which belongs to* $C^d(\mathbf{R})$. *Also,* $T_I$ *is a unitary isomorphism from* $P_I L^2(\mathbf{R})$ *to* $L^2(I)$.

*Proof.* Since $\chi_I U_0 U_1\, f$ satisfies Eq.(5), we have by Eq.(12) that $[W_I^* \chi_I U_0 U_1 f]^{(n)}(\alpha_0+) = [W_I^* \chi_I U_0 U_1 f]^{(n)}(\alpha_1-)$ for all $0 \le n \le d$. Hence $W_I^* \chi_I U_0 U_1 f$ has a unique continuous periodic extension in $C^d(\mathbf{R})$.

For the second part, we note that $U_0 U_1$ is a unitary isomorphism from $U_0^* U_1^* \chi_I L^2(\mathbf{R}) \cong P_I L^2(\mathbf{R})$ to $\chi_I L^2(\mathbf{R})$, and $W_I^* \chi_I$ is a unitary automorphism on $\chi_I L^2(\mathbf{R}) \cong L^2(I)$. □

A natural choice for the periodic basis is $e_{k,j}(t) = \exp(2\pi i j t / |I_k|)$. Another is periodized compactly supported wavelets, which avoid the big support problem of smoother wavelets. Likewise periodic wavelet packets can be used to construct local "best-bases" with no artifacts from cutting or windowing. Discrete versions exist for all of these transforms.

## References


Pascal Auscher, Guido Weiss, and M. V. Wickerhauser, *Local sine and cosine bases of Coifman and Meyer and the construction of smooth wavelets*, Wavelets–A Tutorial in Theory and Applications, C. K. Chui (ed.) ISBN 0-12-174590-2, Academic Press, Boston, 1992, pp. 237–256.

R. R. Coifman and Y. Meyer, *Remarques sur l'analyse de Fourier à fenêtre*, série I, C. R. Acad. Sci. Paris **312** (1991), 259–261.

H. Malvar, *Lapped transforms for efficient transform/subband coding*, IEEE Trans. Acoustics, Speech, and Signal Processing **38** (1990), 969–978.

*Titre en Français:*

**Bases lisses, orthonormales et localizées**

*Version abrégée en Français:*

Commençons avec une fonction $r = r(t)$ de classe $C^d$, $0 \leq d \leq \infty$, qui satisfait la condition (1). Les fonctions qui satisfaient cette condition sont definies en general dans [AWW]. Avec $r$ on definit deux opérateurs $U$ et $U^*$ comme dans les equations (2) et (3). Donc on a la lemme suivante:

**Lemme 1.** *Si $f \in C^d(\mathbf{R})$, donc $Uf \in C^d(\mathbf{R}\backslash\{0\})$, et les limites $[Uf]^{(n)}(0+)$ et $[Uf]^{(n)}(0-)$ existent pour tout $0 \leq n \leq d$. On a:*

$$[Uf]^{(n)}(0+) = 0 \quad \text{si } n \text{ est impair}, \qquad [Uf]^{(n)}(0-) = 0 \quad \text{si } n \text{ est pair}. \tag{4}$$

*Si $f \in C^d(\mathbf{R} \setminus \{0\})$ a des limites $f^{(n)}(0+)$ et $f^{(n)}(0-)$ pour tout $0 \leq n \leq d$ qui satisfont les equations*

$$f^{(n)}(0+) = 0 \quad \text{si } n \text{ est impair}, \qquad f^{(n)}(0-) = 0 \quad \text{si } n \text{ est pair}, \tag{5}$$

*donc $U^*f$ a une extension unique dans la classe $C^d(\mathbf{R})$.* □

Avec les opérateurs de translation et dilatation (definit dans l'equation (6)) on peut ecrire un factorisation (7) de l'operateur lisse de projection $P_{(\alpha_0,\alpha_1)}$, qui est definit dans [AWW]. Ici on ecrit $U_i = U(r, \alpha_i, \epsilon) = \tau^*_{\alpha_i} \delta^*_{\epsilon_i} U(r) \delta_{\epsilon_i} \tau_{\alpha_i}$, $i = 0, 1$. Finalement, on definit deux opérateurs $W, W^*$ par les equations (9), (10). Si $f$ est periodique et satisfait les conditions (5) jusque aux $\alpha_0+$ et $\alpha_1-$, donc l'equation (12) est satisfait, et l'extension périodique de la fonction $W^*_I f$ se trouve dans la classe $C^d(\mathbf{R})$.

Les trois resultats principales sont les suivants:

**Théoreme 2.** *Si $\{e_j\}_{j\in\mathbf{Z}}$ est une collection des fonctions $|I|$-periodiques, qui forment une base orthonormale de $L^2(I)$, donc $E_0 = \{U^*_0 U^*_1 \chi_I W_I\, e_j\}_{j\in\mathbf{Z}}$ est une base orthonormale de $P_I L^2(\mathbf{R})$. De plus, si $\{e_j\}_{j\in\mathbf{Z}} \subset C^d(\mathbf{R})$, donc $E_0 \subset C^d_0(\mathbf{R})$.* □

En decomposant $\mathbf{R}$ par des intervales compatible et adjacent on obtient une base orthonormale de $L^2(\mathbf{R})$.

**Corollaire 3.** *Si* $\mathbf{R} = \bigcup_{k \in \mathbf{Z}} I_k \stackrel{\text{def}}{=} \bigcup_{k \in \mathbf{Z}} [a_k, a_{k+1})$ *et on definit* $U_k^* = U^*(r_k, \alpha_k, \epsilon_k)$ *et* $W_k = W(\tilde{r}_k, I_k, \tilde{\epsilon}_k)$, *et si la famille des fonctions* $|I_k|$*-periodiques* $\{e_{k,j} : j \in \mathbf{Z}\}$ *donnes une base orthonormale de* $L^2(I_k)$, $k \in \mathbf{Z}$, *donc la collection* $E = \{U_k^* U_{k+1}^* \chi_{I_k} W_k \, e_{k,j} : j, k \in \mathbf{Z}\}$ *est une base orthonormale de* $L^2(\mathbf{R})$ *qui consiste des fonctions à support compact. De plus, si toutes les fonctions* $e_{k,j}$ *et* $r_k$, $k, j \in \mathbf{Z}$, *sont dans la classe* $C^d(\mathbf{R})$, *donc les fonctions* $E$ *se trouvent dans* $C_0^d(\mathbf{R})$.

**Theorem 4.** *On ecrit* $I = (\alpha_0, \alpha_1)$, $U_0 = U(r_0, \alpha_0, \epsilon_0)$, $U_1 = U(r_1, \alpha_1, \epsilon_1)$, *et* $W_I^* = W^*(r, I, \epsilon)$. *Si* $f \in C^d(\mathbf{R})$, *donc* $T_I f \stackrel{\text{def}}{=} W_I^* \chi_I U_0 U_1 \, f$ *a une extension* $I$*-periodique dans* $C^d(\mathbf{R})$. *De plus,* $T_I$ *est une isomorphisme unitaire entre* $P_I L^2(\mathbf{R})$ *et* $L^2(I)$.