ELEMENTARY WAVELETS

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Abstract. Necessary and sufficient conditions are given for some functions in $L^2(\mathbf{R})$ with compactly supported Fourier transforms to generate orthonormal bases under the action of integer translations and power-of-two dilations.

Introduction

Some work of A. Grossman and J. Morlet [GM] brought together the interests of engineers and classical harmonic analysts in the construction of "nice" bases for classical function spaces. Seeking square-integrable representations of the $ax+b$ group, they characterized admissible vectors and their orbits. They mentioned the problem of constructing a basis for $L^2(\mathbf{R})$ from an orbit of a discrete subset of the $ax+b$ group. Much of the group theory they would like to use in this situation, however, fails to apply due to the significant differences between the $ax+b$ group and its discrete subsets.

Y. Meyer [M] then found a special function, which he called a "wavelet," smooth and with exponential decay at infinity, whose discrete orbit was a Hilbert basis for $L^2(\mathbf{R})$, and an unconditional basis for almost every common Banach space of functions on $\mathbf{R}$. P. G. Lemarié [L] used this basis to prove certain basic facts about algebras of Calderón-Zygmund operators.

More recently, I. Daubechies [D] has found an algorithm for constructing other wavelets tailored to particular function spaces, including wavelets of compact support.

In this paper, we will explore Meyer-type bases for $L^2(\mathbf{R})$, trying to obtain the simplest possible Hilbert bases with vectors lying in specified "nice" subspaces. Our principal tool will be a unitary isomorphism $\theta : L^2(\mathbf{R}) \to L^2(D \times \mathbf{Z})$, where $D \subset \hat{\mathbf{R}}$ is a dyadic interval together with its reflection about 0. The map θ intertwines the $ax+b$ group with shifts and multiplications by exponentials, providing insight into the orbit through certain simple functions. We shall call $\psi \in L^2(\mathbf{R})$ a *basis wavelet* if its discrete orbit is a Hilbert basis of $L^2(\mathbf{R})$.

We will focus on the simplest types of functions describable by their images under θ. Observe that $L^2(D) \otimes l^2 \hookrightarrow L^2(D \times \mathbf{Z})$. An *elementary tensor wavelet* will be a basis wavelet ψ belonging to $L^2(\mathbf{R})$ such that $\theta\psi$ is an elementary tensor $f \otimes e_m$ in $L^2(D) \otimes l^2$, and a *k-tensor wavelet* will be a basis wavelet ψ for which $\theta\psi$ is a linear combination of k elementary tensors. The main theorems are the uniqueness of the elementary tensor wavelets and a functional equation whose solutions are

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all possible 2-tensor wavelets. Note that Meyer's wavelet, itself a 2-tensor wavelet, is a particular solution to this functional equation.

In a similar fashion, we can obtain necessary and sufficient conditions for the existence of k-tensor wavelets, which exhaust the class of L^2 functions whose Fourier transforms are compactly supported in $\hat{\mathbf{R}} - \{0\}$.

The Unitary Isomorphism

Choose $z > 0$ and let $D = [-2z, -z) \cup (z, 2z] \subset \hat{\mathbf{R}}$. Define $\theta : L^2(\mathbf{R}) \to L^2(D \times \mathbf{Z})$ by

$$\theta f(t,n) = 2^{-n/2} \hat{f}(2^{-n} t) \qquad \text{for } t \in D,\ n \in \mathbf{Z}.$$

Then θ is an isometry, since:

$$\begin{aligned} \|\theta f\|^2 &= \sum_{n \in \mathbf{Z}} \int_D |\theta f(t,n)|^2 dt = \sum_{n \in \mathbf{Z}} \int_D |\hat{f}(2^{-n}t)|^2 2^{-n} dt \\ &= \sum_{n \in \mathbf{Z}} \int_{2^n D} |\hat{f}(t)|^2 dt = \int_{\cup 2^n D} |\hat{f}|^2 dt = \|\hat{f}\|^2 = \|f\|^2. \end{aligned}$$

Also, θ is surjective since for any nonzero $\xi \in \hat{\mathbf{R}}$ there are unique $t \in D$ and $n \in \mathbf{Z}$ such that $\xi = 2^n t$. Hence, if $g \in L^2(D \times \mathbf{Z})$, there is a well defined $f \in L^2$ satisfying $\hat{f}(\xi) = 2^{n/2} g(n,t)$. Evidently, $g = \theta f$.

Finally, θ is unitary because the Fourier transform is unitary. For $f, g \in L^2(\mathbf{R})$, we have:

$$\begin{aligned} \langle \theta f, \theta g \rangle &= \sum_{n \in \mathbf{Z}} \int_D \theta f(t,n) \overline{\theta g(t,n)} dt \\ &= \sum_{n \in \mathbf{Z}} \int_D \hat{f}(2^{-n}t) \overline{\hat{g}(2^{-n}t)} 2^{-n} dt \\ &= \int_{\hat{\mathbf{R}}} \hat{f}(t) \overline{\hat{g}(t)} dt \\ &= \langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle \qquad \text{by Plancherel's Theorem.} \end{aligned}$$

The $ax + b$ group, isomorphic to $\mathbf{R} \ltimes \mathbf{R}$, has the following multiplication law: For $a, b, c, d \in \mathbf{R}$,

$$\begin{aligned} (a,b)(c,d) &= (a + c, 2^c b + d), \text{ and} \\ (a,b)^{-1} &= (-a, -2^{-a} b). \end{aligned}$$

It has a faithful unitary representation U on $L^2(\mathbf{R})$ as the semidirect product of translations and dilations: For $f \in L^2(\mathbf{R})$,

$$U(a,b)f(x) = 2^{a/2} f(2^a x - b).$$

Definition. *Set $L = \{(a,b) | a, b \in \mathbf{Z}\}$. Denote by I those elements of L with nonnegative first coordinate: then $I = \{(a,b) | a, b \in \mathbf{Z}, a \geq 0\}$. Denote by G the group generated by L : then $G = \{(a,b) | a \in \mathbf{Z}, 2^n b \in \mathbf{Z}$ for some $n = n(b) \in \mathbf{Z}\}$.*

G is the group of dyadic rational translations followed by dyadic dilations. L is the set of integer translations followed by dyadic dilations. Both are generated by I.

Lemma 1. *The subset $I \subset G$ is a semigroup.*

Proof. □

Lemma 2. *Every $(x, y) \in G$ may be written as $(x, y) = (a, b)(c, d)^{-1}$, where $(a, b) \in I$ and $(c, d) \in I$.*

Proof. If $y = k2^{-n}$ with $n, k \in \mathbf{Z},\ n \geq 0$, then we may choose $c = max\{n, -x\},\ a = x + c,\ d = 0$, and

$$b = \begin{cases} k, & \text{if } x \geq -n, \\ k2^{x-n}, & \text{if } x < -n. \end{cases} \quad \square$$

Since U is faithful, we shall abuse notation and identify the group G with $U(G),\ L$ with $U(L)$ and I with $U(I)$.

Lemma 3. *If $f \in L^2(\mathbf{R})$ and (a, b) and (c, d) belong to G, then*

$$\langle U(a, b)f, U(c, d)f\rangle = \langle U(a - c, b - 2^{a-c}d)f, f\rangle.$$

Proof. □

This leads to a criterion for orthogonality in an orbit of L :

Proposition 4. *The set $Lf = \{U(a, b)f | a, b \in \mathbf{Z}\}$ is orthonormal in $L^2(\mathbf{R})$ if and only if $\langle U(a, b)f, f\rangle = \delta_e$ for $(a, b) \in I$, where $e = (0, 0)$ is the identity in G.*

Proof. By Lemma 3, we can reduce $\langle U(c, d)f, U(c', d')f\rangle = \langle U(a, b)f, f\rangle$, where $(a, b) = (c - c', d - 2^{c-c'}d)$. Without loss, we can arrange that $c \geq c'$ so that $(a, b) \in I$. But $(a, b) = (0, 0)$ if and only if $c = c'$ and $d = d'$. □

Intertwining L and θ

The map θ is a unitary equivalence between the representation U of L (or G, or I,) on $L^2(\mathbf{R})$ and another representation V on $L^2(D \times \mathbf{Z})$:

Definition. *For $(a, b) \in G$, define $V(a, b) : L^2(D \times \mathbf{Z}) \to L^2(D \times \mathbf{Z})$ by $V\theta f = \theta U f$ for every $f \in L^2(\mathbf{R})$. Then we have, for $\xi = \{\xi_n(t)\} \in L^2(D \times \mathbf{Z})$, the relation*

$$\left(V(a, b)\xi\right)_n(t) = e^{-ibt2^{-a-n}}\xi_{a+n}(t).$$

Since θ is unitary, we have

Lemma 5. *If $\xi = \theta f$, then for all $(a, b) \in G$,*

$$\langle U(a, b)f, f\rangle = \langle V(a, b)\xi, \xi\rangle.$$

Proof. □

Since this holds for $(a, b) \in I$, one can check orthogonality in $L^2(D \times \mathbf{Z})$, which is technically easier.

The Influence of D

Recall that $D = [-2z, -z) \cup (z, 2z]$ for $z > 0$. Of course the choice of z influences θ, but exactly doubling z is a unitary isomorphism:

Lemma 6. *If V and V' are the representations of G on $L^2(D \times \mathbf{Z})$ and $L^2(2D \times \mathbf{Z})$, respectively, given by the previous definition, then V and V' are unitarily equivalent.*

Proof. It suffices to find a unitary isomorphism $\eta : L^2(D\times\mathbf{Z}) \to L^2(2D\times\mathbf{Z})$ such that $V = \eta^{-1}V'\eta$. But η defined as below works:

$$\eta(\xi)_n(t) = 2^{-\frac{1}{2}}\xi_{n-1}(\frac{1}{2}t) \qquad \text{for } t \in 2D.$$

Then for $n \in \mathbf{Z},\ t \in D$, one has:

$$\begin{aligned}
\eta^{-1}V'(a,b)\eta\xi_n(t) &= 2^{\frac{1}{2}}V'(a,b)\eta\xi_{n+1}(2t) \\
&= 2^{\frac{1}{2}}\exp\left(-ib(2t)2^{-a-(n+1)}\right)\eta\xi_{a+n+1}(2t) \\
&= 2^{\frac{1}{2}}\exp\left(-ib(2t)2^{-a-(n+1)}\right)2^{-\frac{1}{2}}\xi_{a+n+1-1}(2(\frac{1}{2}t)) \\
&= \exp(-ibt2^{-a-n})\xi_{a+n}(t) \\
&= V(a,b)\xi_n(t).
\end{aligned}$$

A similarly straightforward calculation shows that $\langle \xi', \eta\xi\rangle_{L^2(2D\times\mathbf{Z})} = \langle \eta^{-1}\xi', \xi\rangle_{L^2(D\times\mathbf{Z})}$ for every $\xi \in L^2(D \times \mathbf{Z}),\ \xi' \in L^2(2D \times \mathbf{Z})$. That η is an isomorphism is clear. □

Corollary 7. *Every representation V' of G is unitarily equivalent to a representation V on $L^2(D\times \mathbf{Z})$ for some $D \subset [-\pi, \pi]$.*

Proof. □

Many tools are available to study the representation V when $D \subset [-\pi, \pi]$.

Elementary Tensor Wavelets

Let $f : D \to \mathbf{C}$ be a fixed function, $f \in L^2(D)$. Let $\xi_n(t) = f(t)e_m(n) \in L^2(D) \otimes l^2$, where

$$e_m(n) = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

This e_m is a unit vector inl^2. Define $\psi = \psi(x),\ \psi \in L^2(\mathbf{R})$, by the equation $\theta\psi = \xi$, namely,

$$(\theta\psi)_n(t) = \begin{cases} 0, & \text{if } n \neq m, \\ f(t), & \text{if } n = m. \end{cases}$$

Then necessary and sufficient conditions for $L\psi$ to be an orthonormal set may be found in terms of f :

Proposition 8. *$L\psi$ is an orthonormal set in $L^2(\mathbf{R})$ if and only if*

$$\int_D e^{-ibt2^{-m}}|f(t)|^2 dt = \delta_0(b) \qquad \text{for } b \in \mathbf{Z}.$$

Proof. By Lemma 5 and Proposition 4, $L\psi$ is an orthonormal set if and only if

$$\langle V(a,b)\xi, \xi\rangle = \delta_e(a,b) \qquad \text{for } (a,b) \in I.$$

With ψ as above, this translates into

$$\begin{aligned} \delta_e(a,b) &= \sum_{n\in\mathbf{Z}} \int_D (V(a,b)\xi)_n(t)\overline{\xi_n(t)}dt \\ &= \sum_{n\in\mathbf{Z}} \int_D e^{-ibt2^{-a-n}} f(t)e_m(a+n)\overline{f(t)e_m(n)}dt \\ &= \begin{cases} \int_D e^{-ibt2^{-m}}|f(t)|^2 dt, & \text{if } a = 0, \\ 0, & \text{if } a \neq 0. \quad \square \end{cases} \end{aligned}$$

Because of Corollary 7 we can assume without loss that $D \subset [-\pi, \pi]$. It is useful to think of f as a function on $[-\pi, \pi]$ (or equivalently, on the circle T) which is zero off D. Then the condition on f in Proposition 8 becomes a condition on the Fourier coefficients of $|f|^2$.

Proposition 9. *If $L\psi$ is an orthonormal set in $L^2(\mathbf{R})$, then $m < 0$.*

Proof. Otherwise, since all Fourier coefficients but the $0th$ vanish, we must have $|f|^2$ = constant almost everywhere in T. But $D^c \cup T$ has positive measure, and $f(t) = 0$ off D forces $f \equiv 0$ and $L\psi = \{0\}$, not an orthonormal set. $\square$

Definition. *For $t \in T \equiv [-\pi, \pi]$, define $(t)_\lambda = \{t, t + \frac{2\pi}{\lambda}, \ldots, t + \frac{(\lambda-1)2\pi}{\lambda}\}$ (mod T), which is the orbit of t under the (natural) action of the $\lambda^t h$ roots of unity.*

Definition. *For $h \in L^2(T)$, define $S_\lambda h(t) = \lambda^{-1}\sum_{s\in(t)_\lambda} h(s)$, which is the symmetrization of h over λ points. Note that $S_\lambda h$ has the λth Fourier coefficients of h, with all of its other Fourier coefficients vanishing:*

$$(S_\lambda h\hat{)}(k) = \begin{cases} \hat{h}(k), & \text{if } \lambda \mid k, \\ 0, & \text{if } \lambda \nmid k. \end{cases}$$

Proposition 10. *If $L\psi$ is an orthonormal set in $L^2(\mathbf{R})$, then $|f|^2$ has the following symmetry property: For almost all $t \in T$,*

$$(*) \qquad \sum_{k=0}^{2^{-m}-1} \left| f(t + \frac{2\pi k}{2^{-m}}) \right|^2 = S_{2^{-m}}|f|^2(t) = 1.$$

Proof. By Proposition 9, $m < 0$. From Proposition 8 we conclude that $(|f|^2\hat{)}(2^{-m}k) = 0$ for all integers $k \neq 0$. But then $(S_{2^{-m}}|f|^2\hat{)}(2^{-m}k) = 0$ for all integers $k \neq 0$. Thus $S_{2^{-m}}|f|^2 \equiv$constant$\neq$ 0, and we may normalize this nonzero constant to 1. $\square$

We now characterize those D for which it is possible to have an elementary tensor wavelet. Observe that if $L\psi$ is orthonormal, then for almost every $t \in T$ at least one summand of equation $(*)$ must be nonzero.

Proposition 11. *If $L\psi$ is a Hilbert basis for $L^2(\mathbf{R})$, then for almost every $t \in T$ at most one summand of equation $(*)$ may be nonzero, and $f(t) \neq 0$ for almost every $t \in D$.*

Proof. To begin with, if f vanishes on a set $P \subset D$ of positive measure, then, the function $g \in L^2(\mathbf{R})$ defined by $\theta g(t,n) = \chi_P(t)e_0(n)$ is nonzero and orthogonal to $L\psi$. This contradicts the assumption that $L\psi$ is a basis.

Further, if more than one summand in equation $(*)$ is nonzero, we can construct another nonzero $g \in L^2(\mathbf{R})$ orthogonal to $L\psi$ because of cancellation. From any subset of D of positive measure on which at least 2 summands of $S_{2^{-m}}|f|^2$ are nonzero, we can choose a measurable subset P satisfying the following:

(i) $|P| > 0$,
(ii) For all $t \in P,\ t + \frac{2\pi k}{2^{-m}} \pmod{T}$ belongs to D for some fixed $k = k(P) \neq 0$,
(iii) Both $|f(t)| > \epsilon$ and $\left|f(t + \frac{2\pi k}{2^{-m}})\right| > \epsilon$ for some fixed $\epsilon > 0$ and all $t \in P$,
(iv) $P \cap \left(P + \frac{2\pi k}{2^{-m}}\right) = \emptyset$.

Define $g \in L^2(D)$ by:

$$\hat{g}(t) = \begin{cases} 1/\bar{f}(t), & \text{if } t \in P, \\ -1/\bar{f}(t), & \text{if } t \in P + \frac{2\pi k}{2^{-m}} \pmod{T}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $S_{2^{-m}}(f\overline{\hat{g}})(t) = 0$, so that for every integer b,

$$\int_D \exp(-ibt2^{-m}) f(t)\overline{\hat{g}(t)}dt = 0.$$

But because of the compact support of $\hat{g}$, we also have the following equation for all integers a, b:

$$\int_D \exp(-ibt2^{-m}) f(t)\overline{\hat{g}(2^{-m-a}t)}dt = 0.$$

Together,these equations imply that $\langle V(a,b)\theta\psi, \theta g\rangle = 0$ for all integers a, b, and thus that $g \perp L\psi$. But by its construction, g is a nonzero element of $L^2(\mathbf{R})$. This contradicts the assumption that $L\psi$ is a basis. □

Proposition 11 imposes strong restrictions on D, and likewise on f.

Theorem 12. *If ψ is an elementary tensor wavelet, then $|\hat{\psi}(t)| = (2\pi)^{-\frac{1}{2}}$ for almost every $t \in [-2\pi, -\pi) \cup (\pi, 2\pi]$, and $\hat{\psi}(t) \equiv 0$ elsewhere.*

Proof. Write $\theta\psi = f \otimes e_m$. By Proposition 9, $m < 0$. By Proposition 11 and the preceding observation, D must have the property that exactly one point of $(t)_{2^{-m}}$ belongs to D for almost every $t \in [-\pi, \pi] = T$. Hence if $t \in D$, then no other point of $(t)_{2^{-m}}$ may be in D, so that T is covered by the 2^{-m} disjoint translates by $2\pi/2^{-m}$ of $D \pmod{T}$, which we may as well call $(D)_{2^{-m}}$. Comparing lengths forces $2^{-m}|D| = |T| = 2\pi \Rightarrow D = [-2\pi 2^m, \pi 2^m) \cup (\pi 2^m, 2\pi 2^m]$, which is easily seen to work.

Now, since exactly one summand of $S_{2^{-m}}|f|^2$ contributes in equation $(*)$, $|f|^2$ must be constant. Recalling that $\theta\psi(t,n) = 2^{-\frac{n}{2}}\hat{\psi}(2^{-n}t)$ yields:

$$\hat{\psi}(t) = \begin{cases} 2^{\frac{m}{2}} f(2^m), & \text{if } 2^m t \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $|\psi|$ must be constant on $2^{-m}D = [-2\pi, -\pi) \cup (\pi, 2\pi]$ and zero elsewhere. Normalizing $\hat{\psi}$ so that $\|\psi\|_{L^2(\mathbf{R})} = 1$ completes the proof. □

This theorem characterizes elementary tensor wavelets. They are the images of the wavelet ψ_o under the group $\mathbf{M}$ of unitary multiplication operators, where $\hat{\psi}_o = (2\pi)^{-\frac{1}{2}}\chi_D$ and $M \in \mathbf{M}$ has the action $(Mf)\hat{}(t) = \mu(t)\hat{f}(t)$ for $|\mu(t)| = 1$ at almost every $t \in \mathbf{R}$. Notice that $\mathbf{M}$ contains all translations as well as the Hilbert transform. Elementary tensor wavelets are unique up to the equivalence defined by $\mathbf{M}$.

It is not hard to compute that $\psi_o(x) = (\sin 2\pi x - \sin \pi x)/(\pi x)$. Such functions have uses in information theory because their frequencies are localized to a single octave. See, for example, the discussion of band-limited functions in McKean [DMcK]. Unfortunately, these wavelets suffer from significant deficiencies. In particular, no elementary tensor wavelet can have a continuous Fourier transform, so it cannot provide a basis for $L^1(\mathbf{R})$ (or $H^1(\mathbf{R})$).

Two-tensor Wavelets

For fixed $j, k \in \mathbf{Z}$, $j < k$, and fixed $f, g \in L^2(D)$, define $\psi \in L^2(\mathbf{R})$ by $\theta\psi = f \otimes e_j + g \otimes e_k$. Then the orthnormality condition may be written as below:

$$\begin{aligned}
\delta_e(a,b) &= \langle U(a,b)\psi, \psi\rangle \qquad \text{for } a, b \in I \\
&= \langle V(a,b)\theta\psi, \theta\psi\rangle \\
&= \sum_{n\in\mathbf{Z}} \int_D \exp(-ibt2^{-a-n})[f(t)e_j(a+n) + g(t)e_k(a+n)]\overline{[f(t)e_j(n) + g(t)e_k(n)]}dt \\
&= \begin{cases} 0, & \text{if } a \neq 0 \text{ and } a \neq k-j, \\ \int_D \exp(-ibt2^{-k})g(t)\overline{f(t)}dt, & \text{if } a = k-j, \\ \int_D [\exp(-ibt2^{-j})|f(t)|^2 + \exp(-ibt2^{-k})|g(t)|^2]dt, & \text{if } a = 0. \end{cases}
\end{aligned}$$

This calculation may be summed up as follows:

Lemma 13. *If $\psi = \theta^{-1}(f \otimes e_j + g \otimes e_k)$ is a 2-tensor wavelet, then*

(1) $\int_D \exp(-ibt2^{-k})g(t)\overline{f(t)}dt = 0$, *and*
(2) $\int_D [\exp(-ibt2^{-j})|f(t)|^2 + \exp(-ibt2^{-k})|g(t)|^2]dt = \delta_0(b)$.

Proof. □

As usual, we assume without loss that $D \subset [-\pi, \pi]$. Lemma 14 is an easy consequence of Lemma 15, but is presented to clarify some of the ideas in the latter's proof.

Lemma 14. *If $\psi = \theta^{-1}(f \otimes e_j + g \otimes e_k)$ is a 2-tensor wavelet, where $j < k$, then $j < 0$.*

Proof. Let $\lambda = 2^{k-j}$. Then $\lambda \geq 2$. The second equation in Lemma 13 implies that $(|f|^2)\hat{}(2^{-j}b) = -\lambda(|g_\lambda|^2)\hat{}(2^{-j}b)$, where $g_\lambda(t) = g(\lambda t)$ and $b \neq 0$. But then, if $j \geq 0$, this equation relates all but the the $0th$ Fourier coefficients of $|f|^2$ and $|g_\lambda|^2$. Without loss, we can write

$$|f|^2 + \lambda|g_\lambda|^2 = 1.$$

Now f and g_λ have disjoint support, showing immediately that $|f|^2 = \chi_D$ and $|g|^2 = \lambda^{-1}\chi_D$. But the first equation of Lemma 13 shows that

$$(g\bar{f})\hat{}(2^{-k}b) = 0 \qquad \text{for all } b \in \mathbf{Z}.$$

Again, if $k > j \geq 0$, this implies that $g\bar{f} \equiv 0$ and that $|g|^2|f|^2 \equiv 0$, which contradicts $|g|^2|f|^2 = \lambda^{-1}\chi_D$. □

Lemma 15. *If $\psi = \theta^{-1}(f \otimes e_j + g \otimes e_k)$ is a 2-tensor wavelet, where $j < k$ and $g \not\equiv 0$, then $k < 0$.*

Proof. Suppose that $k \geq 0$. Then $n2^k \in \mathbf{Z}$ for all $n \in \mathbf{Z}$. From the second equation of Lemma 13 with $b = n2^k$ and setting $\lambda = 2^{k-j} \geq 2$, one obtains

$$(**) \qquad (|g|^2\hat{)}(n) + (|f|^2\hat{)}(\lambda n) = \delta_0(n) \qquad \text{for } n \in \mathbf{Z}.$$

This relates all the Fourier cofficients of $|g|^2$ to those of $|f|^2$ which are multiples of λ. But these latter coefficients may be isolated by using the symmetrization operator S_λ :

$$(|f|^2\hat{)}(\lambda n) = (S_\lambda|f|^2\hat{)}(\lambda n).$$

Recalling the notation $h_\lambda(t) = h(\lambda t)$ and using a well known property of the Fourier transform yields:

$$(|g|^2\hat{)}(n) = (\lambda|g_\lambda|^2\hat{)}(\lambda n).$$

Putting these two equations together with equation $(**)$ yields

$$(S_\lambda|f|^2\hat{)}(\lambda n) + (\lambda|g_\lambda|^2\hat{)}(\lambda n) = \delta_0(\lambda n) \qquad \text{for all } n \in \mathbf{Z}.$$

All other Fourier coefficients in the above equation vanish, so it is possible to conclude that:

$$S_\lambda|f|^2(t) + \lambda|g_\lambda|^2(t) = 1 \qquad \text{for all } t \in T.$$

Now $|g_\lambda|^2(t) = 0$ if $t \in (\lambda^{-1}D)^c$, since supp $g \subset D$. Thus $t \in (\lambda^{-1}D)^c \Rightarrow S_\lambda|f|^2(t) = 1$. But since $S_\lambda|f|^2(t)$ is an average over $(t)_\lambda$, and $r \in (t)_\lambda \Rightarrow (r)_\lambda = (t)_\lambda$, we have that if $t \in (\lambda^{-1}D)^c$ and $r \in (t)_\lambda$, then $S_\lambda|f|^2(r) = 1$ and $g_\lambda(r) = 0$. With $\lambda \geq 2$, it can be shown that for almost all $r \in \lambda^{-1}D$ there exists $t \in (\lambda^{-1}D)^c$ with $r \in (t)_\lambda$. This is because both $r + \pi \in (r)_\lambda$ and $r - \pi \in (r)_\lambda$, and with $\lambda^{-1}D \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$ we are guaranteed that one of $r \pm \pi$ must be in $(\lambda^{-1}D)^c$ for almost every $r \in \lambda^{-1}D$. Thus $|g_\lambda|^2 = 0$ almost everywhere in $\lambda^{-1}D$, so $g = 0$ in $L^2(D)$, contradicting the hypothesis. □

The lemmas above are results of the orthogonality condition for wavelets. They may be combined with the density requirement to yield the main theorem:

Theorem 16. *Suppose $\psi = \theta^{-1}(f \otimes e_j + g \otimes e_k)$ is standardized with $j < k$, f and g different from 0, and $D \subset [-\pi, \pi] \equiv T$. Write $\lambda = 2^{k-j} \geq 2$, $\mu = 2^{-j}$, and $\nu = 2^{-k}$. Then ψ is a 2-tensor wavelet if and only if all of the following are true:*

1. *$j < k < 0$, so that $\mu \geq 2\nu \geq 4$ and $\mu \geq 2\lambda \geq 4$,*
2. *$S_\mu(|f|^2 + \lambda|g_\lambda|^2) \equiv 1$ on T,*
3. *$S_\nu(g\bar{f}) \equiv 0$ on T,*
4. *If $h \in L^2(\mathbf{R})$, then $(\forall n \in \mathbf{Z})\Big(S_\mu([f + \lambda^{\frac{1}{2}}g_\lambda]h_{2^n}) \equiv 0 \text{ on } T\Big) \Leftrightarrow h \equiv 0$.*

Proof. Lemma 15 shows the necessity of (1). We can rewrite the two necessary conditions of Lemma 13 as follows:

$$\int_D \exp(-ibt\nu)g(t)\bar{f}(t)dt = 0 \qquad \text{for all } b \in \mathbf{Z}, \text{ and}$$

$$(***) \qquad \int_D [\exp(-ibt\mu)|f(t)|^2 + \exp(-ibt\mu\lambda)|g(t)|^2]dt = \delta_0(b)$$

$$= \int_{D \cup \lambda^{-1}D} \exp(-ibt\mu)[|f(t)|^2 + \lambda|g_\lambda(t)|^2]dt.$$

With λ and μ as given, these equations in the S_μ notation are equivalent to (2) and (3). Conversely, if $j < k < 0$, then (2) and (3) are equivalent to equations (∗∗∗), which imply that $L\psi$ is an orthonormal set.

Given (1), the condition that $L\psi$ is dense in $L^2(\mathbf{R})$ is equivalent to (4) by the following calculation: For $\phi \in L^2(\mathbf{R})$,

$$\begin{aligned}
\phi = 0 \Leftrightarrow \phi \perp L\psi& \\
\Leftrightarrow 0 &= \langle U(a,b)\psi, \phi\rangle \qquad \text{for all } (a,b) \in L \\
\Leftrightarrow 0 &= \langle V(a,b)\theta\psi, \theta\phi\rangle \qquad \text{for all } (a,b) \in L \\
\Leftrightarrow 0 &= \sum_{n\in\mathbf{Z}} \int_D \exp(-ibt2^{-n-a})\,[e_j(n+a)f(t) + e_k(n+a)g(t)]\,\overline{\theta\phi(t,n)}dt.
\end{aligned}$$

All but two of the summands vanish, and this last condition is equivalent to:

$$0 = \int_D \exp(-ibt2^{-j})f(t)\overline{\theta\phi(t,j-a)}dt + \int_D \exp(-ibt2^{-k})g(t)\overline{\theta\phi(t,k-a)}dt.$$

Now let $h(t) = \overline{\theta\phi(t,j-a)}$. Then $\overline{\theta\phi(t,k-a)} = h(\lambda^{-1}t)$, where we must think of h as a function on T rather than as a function on D. Then $\phi = 0 \Leftrightarrow (\forall n \in \mathbf{Z})\ h_{2^n} \equiv 0$ on T. Changing variables in the second integral and using μ and λ, one obtains the equivalent condition:

$$\begin{aligned}
\phi = 0 \Leftrightarrow 0 &= \int_D \exp(-ibt\mu)f(t)h_{2^n}(t)dt + \\
&\quad + \int_D \exp(-ibt\mu\lambda)g(t)h_{2^n}(\lambda^{-1}t)dt \qquad \text{for all } n \in \mathbf{Z}, \\
\Leftrightarrow 0 &= \int_D \exp(-ibt\mu)f(t)h_{2^n}(t)dt + \\
&\quad + \int_{\lambda^{-1}D} \exp(-ibt\mu)\lambda^{-\frac{1}{2}}g(\lambda t)h_{2^n}(t)dt \qquad \text{for all } n \in \mathbf{Z}, \\
\Leftrightarrow 0 &= S_\mu([f + \lambda^{\frac{1}{2}}g_\lambda]h_{2^n}) \qquad \text{for all } n \in \mathbf{Z}. \quad \Box
\end{aligned}$$

From these functional equations (2)–(4), it is possible to construct examples of basis wavelets. Similar equations may be found for k-tensor wavelets, although they become unwieldy.

Theorem 16 characterizes those D for which one may have 2-tensor wavelet. Since $S_\mu(|f|^2 + \lambda|g_\lambda|^2)(t) \equiv 1$ on T, it is necessary that for almost every $t \in T$ there is some $s \in (t)_\lambda$ such that $s \in D \cup \lambda^{-1}D$. This may be stated as $T \subset (D \cup \lambda^{-1}D)_\mu$. However, condition (4) may hold even if several points of $(t)_\lambda$ belong to $D \cup \lambda^{-1}D$ for a nonnull set of points $t \in D \cup \lambda^{-1}D$. This is because of the possibility that the only nonzero function ϕ which can satisfy $\phi \perp L\psi$ by exploiting the resulting cancellation might not be in $L^2(\mathbf{R})$. In fact, the Fourier transform of Meyer's wavelet is supported in $D \cup 2^{-1}D$, where $\mu = 4,\ \lambda = 2$, and

$$D = [-\frac{2\pi}{3}, -\frac{\pi}{3}) \cup (\frac{\pi}{3}, \frac{2\pi}{3}].$$

This particular D has the property that for every $t \in T$ there are exactly two points in $(t)_\mu$ which belong to $D \cup \lambda^{-1}D$, and either they both belong to D or they both belong to $\lambda^{-1}D$. Call these two points t_1 and t_2 if they belong to D. In addition, $\lambda^{-1}t_1$ and $\lambda^{-1}t_2$ are the only points in $D \cup \lambda^{-1}D$ which are also in $(\lambda^{-1}t_1)_\mu$ or $(\lambda^{-1}t_2)_\mu$. Thus, to satisfy condition (4) for every $n \in \mathbf{Z}$, it is necessary that all three of the equations below hold simultaneously for all pairs $\{t_1, t_2\} = (t)_\mu \cap [D \cup \lambda^{-1}D]$ arising from $t \in D$:

$$f(t_1)\bar{g}(t_1) + f(t_2)\bar{g}(t_2) = 0$$
$$f(t_1)h_{2^n}(t_1) + f(t_2)h_{2^n}(t_2) = 0$$
$$g(t_1)h_{2^n}(t_1) + g(t_2)h_{2^n}(t_2) = 0$$

It is easy to choose functions f and g such that this can only happen if $h \equiv 0$.

In a forthcoming article, we will discuss other limitations on D, μ, and λ imposed by Theorem 16.

References

[D] I. Daubechies, *Discrete series of coherent states and their use in signal analysis*, preprint, Courant Institute of Mathematical Sciences, New York.

[GM] A. Grossman and J. Morlet, *Decomposition of Hardy functions into square-integrable wavelets of constant shape*, SIAM J. Math. Anal. **15** (1984), 723–736.

[L] P. G. Lemarié, *Continuité sur les espaces de Besov des opérateurs définis par les intégrales singulières*, Annales de l'Institute Fourier (to appear).

[M] Y. Meyer, *Principe d'incertitude, bases hilbertiennes et algèbres d'operateurs*, Séminaire Bourbaki **662** (February, 1986).

[DMcK] H. Dym and H. P. McKean, *Fourier series and integrals*, Academic Press, New York, 1972.

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