FAST APPROXIMATE Karhunen–Loève EXPANSIONS

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NOTATION

Let $X = \{X_n : n = 1, \ldots, N\} \in \mathbf{R}^d$ be an ensemble of vectors. We suppose that d is very large and that X spans $\mathbf{R}^d$, implying $N \geq d$. The Karhunen–Loève basis for the ensemble consists of the eigenvectors of the symmetric positive definite autocovariance matrix $M = E(X \otimes X)$, or

$$M_{ij} = \frac{1}{N} \sum_{n=1}^{N} X_n(i) X_n(j).$$

It is known that coefficients with respect to the Karhunen–Loève basis are independent random variables, and that they achieve the maximum linear transform coding gain or equivalently, the minimum entropy of any linear code used to transmit X.

Write $\overline{X} = E(X) = \frac{1}{N} \sum_{n=1}^{N} X_n$, and let $\sigma(X) \in \mathbf{R}^d$ be the vector of variances of the coefficients of X. Namely,

$$\sigma(X)(i) = \frac{1}{N} \sum_{n=1}^{N} [X_n(i) - \overline{X}(i)]^2.$$

We may assume without loss that $\overline{X} = 0$. Write $\mathrm{Var}(X)$ for the sum of the coefficients of $\sigma(X)$, which is the total variance of the ensemble X.

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Let $U : \mathbf{R}^d \to \mathbf{R}^d$ be orthogonal and write $Y = UX$ for the map $Y_n = UX_n$. Since U is linear, $\overline{Y} = \overline{UX} = U\overline{X} = 0$, and since U is orthogonal, $\mathrm{Var}(X) = \mathrm{Var}(Y)$. Define the *transform coding gain* as in [J] by the formula $G_{TC}(U) = \mathrm{Var}(UX)/\exp H(UX)$, where

$$H(X) = \frac{1}{d}\sum_{i=1}^{d} \log \sigma(X)(i).$$

$G_{TC}(UX)$ is maximized when $H(UX)$ is minimized, and H is the entropy of the direct sum of d independent Gaussian random variables with variances $\sigma(X)(i)$, $i = 1, \ldots, d$. The Karhunen–Loève transform is a global minimum for H, and we will say that the best approximation to the Karhunen–Loève transformation from a library $\mathcal{U}$ of orthogonal transformations is the minimum of $H(UX)$ with the constraint $U \in \mathcal{U}$.

Algorithm

Notice that H is an information cost function in the sense of [CMQW]. We may create a large library of orthogonal bases by recursive quadrature mirror filter convolution-decimation, and use the best-basis search algorithm with H to find the best approximation to the Karhunen–Loève basis. In the case $d = 8$ we have:

$X_i(1)$	$X_i(2)$	$X_i(3)$	$X_i(4)$	$X_i(5)$	$X_i(6)$	$X_i(7)$	$X_i(8)$
s_1	s_2	s_3	s_4	d_1	d_2	d_3	d_4
ss_1	ss_2	ds_1	ds_2	sd_1	sd_2	dd_1	dd_2
sss_1	dss_1	sds_1	dds_1	ssd_1	dsd_1	sdd_1	ddd_1

Figure 1. Complete wavelet packet expansion of X_i.

Then the sums of the squares are accumulated in an array of variances:

$\sum_i X_i^2(1)$	$\sum_i X_i^2(2)$	$\sum_i X_i^2(3)$	$\sum_i X_i^2(4)$	$\sum_i X_i^2(5)$	$\sum_i X_i^2(6)$	$\sum_i X_i^2(7)$	$\sum_i X_i^2(8)$
$\sum_i s_1^2$	$\sum_i s_2^2$	$\sum_i s_3^2$	$\sum_i s_4^2$	$\sum_i d_1^2$	$\sum_i d_2^2$	$\sum_i d_3^2$	$\sum_i d_4^2$
$\sum_i ss_1^2$	$\sum_i ss_2^2$	$\sum_i ds_1^2$	$\sum_i ds_2^2$	$\sum_i sd_1^2$	$\sum_i sd_2^2$	$\sum_i dd_1^2$	$\sum_i dd_2^2$
$\sum_i sss_1^2$	$\sum_i dss_1^2$	$\sum_i sds_1^2$	$\sum_i dds_1^2$	$\sum_i ssd_1^2$	$\sum_i dsd_1^2$	$\sum_i sdd_1^2$	$\sum_i ddd_1^2$

Figure 2. Variances of wavelet packet coordinates of X.

This tree of variances may be searched for the orthogonal basis which minimizes H. Let U be this basis, and write $\{U_i \in \mathbf{R}^d : i = 1, \ldots, d\}$ for the rows of U. We may suppose that these rows are numbered so that $\sigma(UX)$ is in decreasing order. Then we fix $\epsilon > 0$ and let d' be the smallest integer such that $\sum_{n=1}^{d'} \sigma(UX)(n) \geq (1-\epsilon)\mathrm{Var}(X)$. Then the projection of X onto the span of $U' = \{U_1 \ldots U_{d'}\}$ contains $1-\epsilon$ of the variance.

Statement of the Main Results

Call the orthogonal projection U' (associated to ϵ) the approximate Karhunen–Loève transform with relative variance error ϵ.

Already these d' vectors U' are a good basis for the ensemble X, but they may be further decorrelated by Karhunen–Loève factor analysis. The algorithm is fast because we expect that even for small ϵ we will obtain $d' \ll d$. Counting operations:

(1) Expanding N vectors $X_n \in \mathbf{R}^d$ into wavelet packet coefficients: $O(Nd\log d)$.

(2) Summing squares into the variance tree: $O(d \log d)$.

(3) Searching the tree for a best basis: $O(d)$.

(4) Sorting the best basis vectors into decreasing order of importance: $O(d \log d)$.

(5) Transforming $U'X$ by Karhunen–Loève: $O({d'}^3)$.

Indeed, the last step may not be necessary, since a large reduction in the number of parameters is already achieved by the orthogonal projection U'.

Applications to the Mug's Gallery Problem

Lawrence Sirovich provided 143 digitized 128x128x8bit pictures of Brown University students. These were already normalized with the pupils impaled on two fixed points near the center of the image. We first transformed the data to floating point numbers, computed average values for the pixels, and subtracted the average from each pixel to obtain “caricatures,” or deviations from the average.

Each caricature was treated as a picture and expanded into 2 dimensional wavelet packets as described in [W]. The squares of the amplitudes were summed into a tree of variances, which was searched via the best-basis search procedure. Call this most-concentrated basis the *joint best basis* for the ensemble. In the joint best basis, 400 coordinates (of 16384) contained more than 90% of the variance of the ensemble. Figure 1 shows the total variance on the first d' coordinates in the joint best basis, sorted in decreasing order, as a fraction of the total variance of the ensemble, for $1 \leq d' \leq 1024$. Using 1024 parameters captures more than 95% of the ensemble variance, but requires somewhat more computer power than is readily available on a desktop. A 400 parameter system can be analyzed on a common workstation in minutes. The top 400 coordinates were recomputed for each caricature and their autocovariance matrix over the ensemble was diagonalized by the LINPACK singular value decomposition routine.

Figure 2 shows the total variance on the first d' coordinates in the Karhunen–Loève basis, sorted in decreasing order, as a fraction of the total variance of the 400 joint best basis coefficients, for $1 \leq d' \leq 143$. The Karhunen–Loève post-processing for this small ensemble concentrates 98% of the retained variance from the top 400 joint best-basis parameters into 10 coefficients.

Figure 3 shows the top 6 Karhunen–Loève "eigenfaces" with respect to the top 400 joint best-basis parameters. These have been normalized to fill the dynamic range of the printing device. Figure 4 shows the top 6 joint best basis wavelet packets. Whereas the Karhunen–Loève basis functions look more or less like faces (or at least heads), the wavelet packets are abstract blobs which can be better localized at specific facial features.

Conclusions

Wavelet packet analysis reduces the number of parameters needed to perform approximate Karhunen–Loève expansions. For a factor analysis "explaining" all but ϵ of the ensemble variance in a d parameter system, the complexity will be $O(d^2 \log d + {d'}^3)$, where $d' \ll d$. For accuracies of 1 or 2 significant digits, the analysis of systems of 16384 paramters can be performed on desktop computers in minutes.

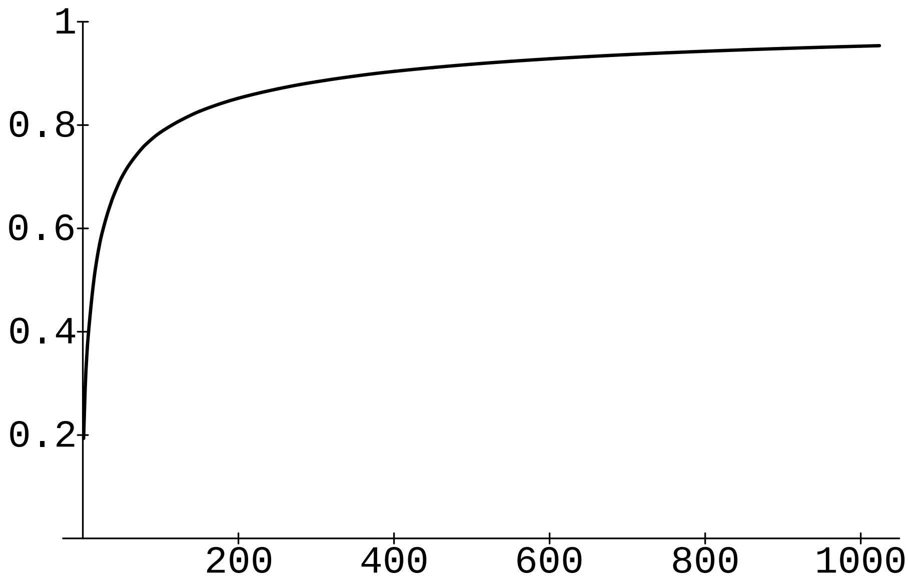

Figure 1. Fraction of the variance in the joint best basis.

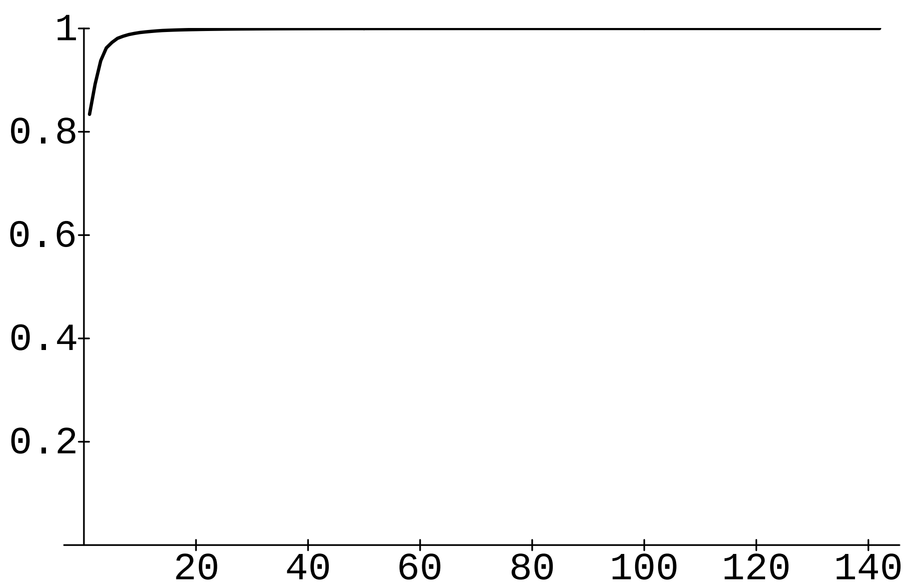

Figure 2. Fraction of the variance in the Karhunen–Loève basis.

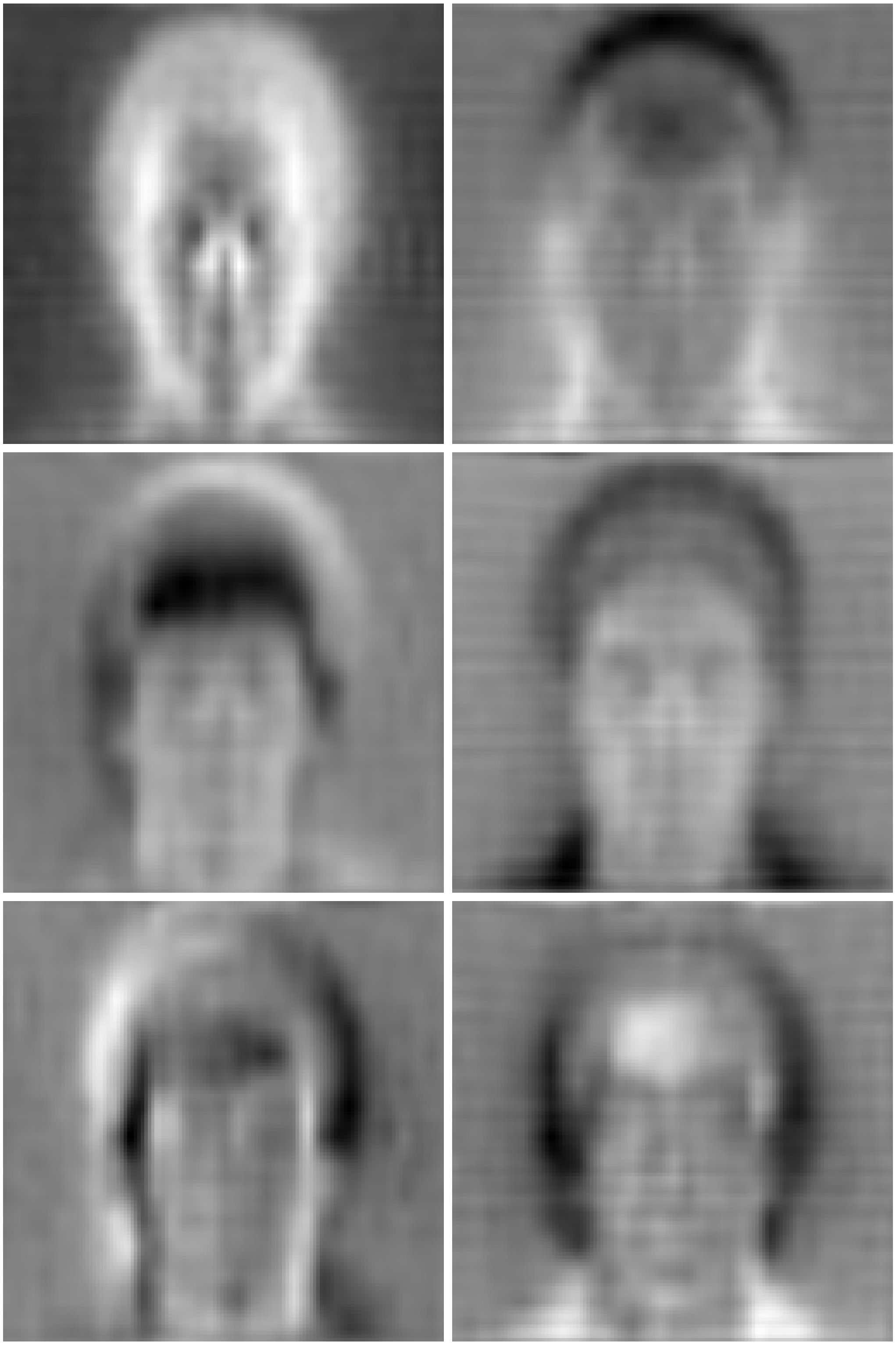

Figure 3. Top six Karhunen–Loève eigenfaces.

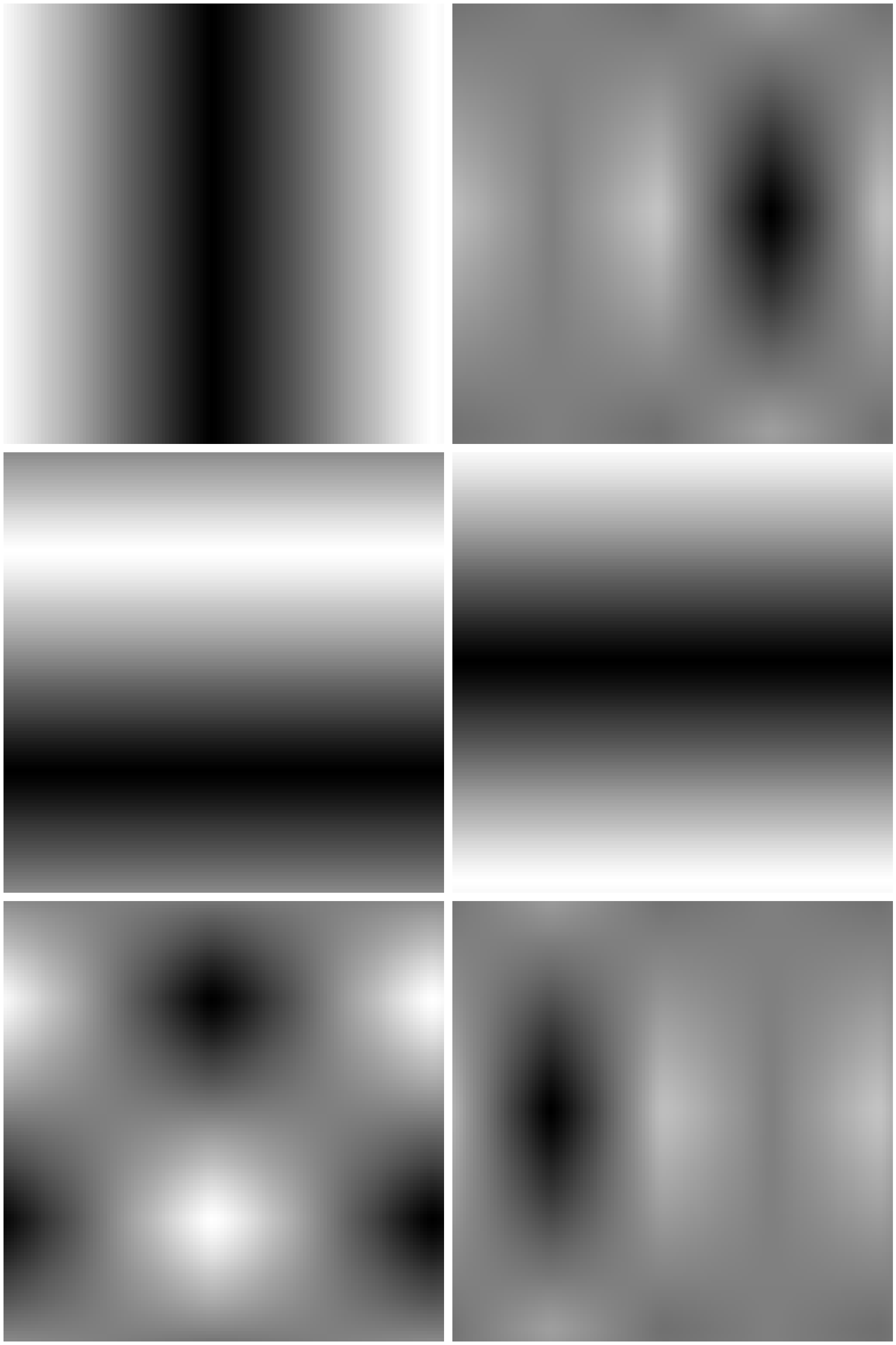

Figure 4. Top six joint best basis functions.

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