Introduction to Section IX: Selected Applications

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1 The Selections

Over the past decade, wavelet transforms have been widely applied. Good implementations of the discrete wavelet transform (DWT) were built into software systems such as Matlab and S-Plus, and DWT became a frequently-used tool for data analysis and signal processing. There are certain problems, though, on which this tool works particularly well. The most common ingredient in those problems is some complicated object that can be closely approximated by a few superposed wavelets. This compilation includes four seminal articles that introduced some of these stand-out DWT applications. I have taken a random and sparse sampling of relevant articles and books published around the same time, in order to place the results in context and illustrate their influence.

2 Fast Evaluation of Singular Integral Operators

Discrete wavelet transforms are fast algorithms, costing $O(1)$ arithmetic operations per output regardless of the number of inputs. This is even better than the fast Fourier transform, which costs $O(\log N)$ operations per output given N inputs, as much as a complete wavelet packet or multiscale local cosine analysis. A general linear transformation of N inputs, by contrast, costs $O(N)$ operations per output. It was seen right away that DWTs could be advantageous in high-dimensional problems.

An example is the evaluation of an integral operator $f \mapsto Tf$ with

$$Tf(x) = \int t(x,y)f(y)\,dy,$$

where t is a smooth function except on some thin subset of its domain. The gravitational potential, with $t(x,y) = 1/|x-y|$, is one such operator. To simulate the time evolution of a many-particle system interacting by gravitation requires repeated recomputation and then evaluation of T. A great deal of work in the 1980s [4, 6, 66] culminated in V. Rokhlin's fast multipole algorithm [32, 17].

The seminal 1991 article in this compilation, "Fast Wavelet Transforms and Numerical Algorithms I" by Beylkin, Coifman and Rokhlin, shows that the fast multipole hierarchical decomposition is in essence a multiresolution analysis. It may be performed fast by an orthogonal pair of conjugate quadrature (mirror) filters. Sparsity of the resulting matrix is guaranteed for Calderon–Zygmund singular integral operators [52] such as the gravitational potential, if the underlying wavelets representing the operator have many vanishing moments.

Subsequently, more complex wavelet-like transforms were brought to bear on ever nastier linear operators to get sparse matrix approximations [3, 7, 2, 65, 53]. Sparse matrix multiplication makes linear algebra feasible even in very high dimensions. The wide class of operators that reduce to sparse matrices in wavelet bases made possible fast algorithms for such difficult problems as numerical homogenization [14], electromagnetic scattering [55], general trigonometric approximation [8], Hilbert transforms [11, 12], The special properties of wavelets also permit linear superpositions to be used in nonlinear functions [22, 10, 9]

Multiresolution decomposition into wavelets with many desirable analytic properties has provided an elegant path into operator theory. The existence of fast discrete wavelet transforms has made this a smooth path to efficient numerical methods as well.

3 Improved Transform Coding Image Compression

Digital images also have the potential to be enormously complicated, but when they are pictures of interest to humans they must actually be relatively simple. Among many techniques for efficient storage or transmission of such pictures is *transform coding image compression* [64]. The Joint Photographic Experts Group (JPEG) algorithm [39, 40, 63] is perhaps the most common, since it is used in the JPG files found throughout the World Wide Web. But JPEG is an approximation algorithm. The errors it introduces, while nearly invisible to the eye, interfere with edge detection and similar image analysis.

The advantages of wavelets are nicely explained in Devore, Jawerth and Lucier's foundation article, "Image Compression Through Wavelet Transform Coding" [24], which is reprinted in this compilation. The absence of JPEG's block artifacts allows compressed images to be used for automatic fingerprint identification systems, and so the United States Federal Bureau of Investigation (FBI) and Great Britain's Scotland Yard collaborated to design a custom wavelet and scalar quantization (WSQ) image compression standard [36, 13]. This relied on symmetric biorthogonal wavelets that were verified as suitable for high resolution images [18, 51], and for which a convenient boundary treatment existed [15].

Subsequently, a more efficient implementation of the biorthogonal wavelet transform used in WSQ was found by Sweldens [61]. Also, redundancy removers that partitioned wavelet coefficients into hierarchical subsets called zero-trees [57, 56] were matched to this family of transforms, producing a remarkably simple and efficient coder. The result became a new standard, called JPEG-2000 [41].

There are other boundary treatments using wavelets on intervals [19], more general transforms such as wavelet packets [21], lapped orthogonal transforms [48], and multiwavelets [60], plus various methods for progressive transmission and error correction coding of wavelet-compressed images that are making their way into proprietary, state-of-the-art coders for pictures and video. It is safe to say that every advantage of wavelets will be exploited in the fierce competition for better image quality and coding efficiency.

4 Easy Generic "De-Noising"

Digitally sampled signals that vary smoothly with time appear rough and may be hard to detect when measurement errors are present in each sample. The model of identically distributed independent normal errors, or "additive Gaussian white noise," is an extreme case of rough noise that is frequently used in practice. There are classical digital signal processing algorithms (DSP) to compute Gaussian white noise power, based on the discrete Fourier transform (DFT). With knowledge of the signal, we may design matched filters in the frequency domain and obtain minimax linear estimators for signal detection [37, 67]

But there are examples where the signal to be detected contains added noise that is correlated in time from sample to sample. Such noise may be smoother than Gaussian white noise, though still rougher than the signal. In addition, the signal itself or even its smoothness may be unknown. In these cases, very complicated estimators have been devised [59].

A much simpler way to build estimators was described in Donoho and Johnstone's "Adapting to unknown smoothness by wavelet shrinkage" [28], which is reprinted in this compilation. It followed a number of papers [25, 26, 27] on the remarkable properties of wavelet coefficient thresholding, the bounded nonlinear operation of reducing or removing small-amplitude wavelet components of a noisy signal.

In practice, there still remains the problem of setting a threshold for wavelet shrinkage. There is a universal value that depends on the noise power, and there are techniques to adjust for correlated noise [44]. When the signal to be detected is known, there is the oracle method which selects a threshold to minimize the estimator variance [20, 16, 23].

Other wavelet transforms, principally the continuous wavelet transform [62], have also found use in signal estimation and detection. Examples include speech and music [47, 33], NMR spectra [34], and even gravitational waves [38].

5 Roughness, Volatility, and Turbulence

How do we estimate the roughness of a continuous function? One way is to calculate the Hölder exponent at each point. Jaffard's seminal 1989 paper "Exposants de Hölder en des Points Donnés et Coéfficients d'Ondelettes" [42], included in this compilation, describes an elegant way to estimate the exponents from the asymptotic decay of wavelet coefficient amplitudes as scale tends to zero. The slower the decay, the smaller the exponent and the rougher the function. This fact leads to an elegant proof [35, 43] of Gerver's famous result on the almost nowhere differentiability of Riemann's function [31].

For a more detailed analysis of roughness, we may inquire about the distribution of Hölder exponents over the domain of a function. The *singularity spectrum* is one way to describe this distribution; it gives the fractal dimension [49] of domain subsets where the function has a particular Hölder exponent. This spectrum is useful in distinguishing physical phenomena [58], and it can be computed efficiently from time series using DWT [54]. The asymptotic behavior of wavelet coefficients can also be used to detect *fractional Brownian motion*, or to synthesize examples [30, 1] which are used in mathematical finance [50].

When a theory predicts a certain degree of roughness, wavelet coefficient asymptotics may be used to test it. Kolmogorov's famous $-5/3$ power law for the velocity power spectrum in fully developed turbulence [45, 46] may be tested this way. We may also compute the singularity spectrum of portions of simulated or measured flows to determine if they are turbulent or laminar [5, 29, 68], for example.

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