Rotational Wavelet Transforms for Motion Analysis, Estimation and Tracking. *

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Abstract

This paper addresses the problem of estimating, analyzing and tracking objects moving with spatio-temporal rotational motion (i.e. the angular velocity of spinning or orbiting motions). It is assumed that the digital signals of interest are acquired from a camera and structured as digital image sequences. The trajectories in the signal are two-dimensional spatial projections in time of motion taking place in a three-dimensional space. The purpose of this work is to focus on the rotational motion i.e. estimate the angular velocity and acceleration. In natural scenes, rotational motion usually evolves on a trajectory and then composes with translational or accelerated motion. This paper shows that the trajectory parameters and the rotational parameters can be efficiently estimated and tracked either simultaneously or separately. The final goal of this work is to provide selective reconstructions of moving objects of interest. This paper constructs new continuous wavelet transforms that can be tuned to both translational and rotational motion. The parameters of analysis that are taken into account in these rotational wavelet transforms are space and time position, velocity, spatial scale, angular orientation and angular velocity. The continuous wavelet functions are finally discretized for signal processing. The link between rotational motion, symmetry and critical sampling is also presented. Simulations have been performed with estimation, detection and tracking on natural scenes.

1 Introduction

In this paper, we present new spatio-temporal continuous wavelet transforms that are maximum likelihood estimators of rotational and translational motion parameters. These wavelets extend our previous work done on the Galilean wavelet transforms dedicated to translational velocity and acceleration [9]. The analysis that we propose in this paper is performed according to criteria based on spatial and temporal position, angular orientation, velocity, angular velocity and spatial scale. The definition of a rotational motion depends upon the axis around which rotation takes place. If the axis is the center of gravity of the object, the motion refers to a *spin* (ball, whirl). If the objet revolves around an external axis, the motion refers to an *orbit.* This paper will mainly focus on processing digital image sequences i.e. two-dimensional space and time signals acquired by a camera or a planar sensor. Some principles about sampling and symmetries will be clearly evidenced in this paper from Fourier spectra and Lie group theory.

The approach of motion filtering considered in this paper differs fundamentally from other techniques that have been proposed in the literature such as those based on optical flow, pel-recursive, block matching and Bayesian models. The continuous wavelet transform provides motion estimations that are robust not only against image noise and blurr but also against motion noise (i.e. jitter) [3], [7], [8]. Moreover, as a result of both the spatio-temporal filtering and the interpolation wavelet properties, the wavelet technique can resolve temporary occlusion problems. It has been demonstrated that the wavelet transform behaves as a matched filter and performs minimum-mean-square error estimation of the motion parameters [7] [8].

The study of the rotational motion is part of the harmonic oscillator. The study of the translational or linear motion belongs to the Galilean wavelet. Translational motion composes with the rotational motion to put the harmonic oscillator on a carrier trajectory (i.e. spinning or orbiting on the carrier trajectory). Fourier

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analysis shows that both motions keep distinct signatures and that trajectories can be built independently from the spinning motion whose estimation can be addressed afterwards. A tracking algorithm is eventually presented that exploits the Lie algebra of a harmonic oscillator, a Kalman filter, the Galilean and the rotational wavelet transforms separately.

2 Rotational Motion and Symmetries

The goal is first to show intuitively that the spectrum of translational and rotational motion are significantly apart. Consequently, translational motion $\vec{v}$ can be independently estimated from the rotational motion θ. This leads to the important fact that the carrier trajectory may be detected, estimated and tracked first and then the existence of angular velocity can be evidenced. The spectrum of the translational motion occupies a particular part of the Fourier domain [3], it is located in a plane orthogonal to the velocity vector $(\vec{v}, 1)$ of equation $\omega + \vec{k} \cdot \vec{v} = 0$.

Let us consider rotational motion in two-dimensional space and time $\mathbf{R}^2 \times \mathbf{R}$. Let $f(\vec{x})$ of space $\mathbf{R}^2$ be a function. If the function is orbiting around the origin with a uniform angular velocity θ. It reads

$$\tilde{f}(\vec{x}, t) = f[R(\theta t)\vec{x}] \tag{1}$$

The rotational transformation is a unitary transformation given by the real rotation

$$R(\theta t) = \begin{pmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{pmatrix} \tag{2}$$

Let us now consider the Fourier transform of the moving function $\tilde{f}(\vec{x}, t)$ in polar coordinates, it is characterized by Hankel transforms. We have indeed

$$\begin{aligned}
\hat{\tilde{f}}(\vec{k}, \omega; \theta_1) &= \int_{\mathbf{R}^2 \times \mathbf{R}} dt\, d^2\vec{x}\, f[R(\theta_1 t)\vec{x}]\; e^{-i(\vec{k}\cdot\vec{x} + \omega t)} \\
\hat{\tilde{f}}(k, \kappa, \omega; \theta_1) &= \int_0^{+\infty} r dr \int_0^{2\pi} d\phi \int_0^{2\pi} dt\; f(re^{i\phi}) \\
&\quad e^{-i[\omega t - kr sin(\theta_1 t - \kappa + \phi)]} \\
&= \int_0^{2\pi} d\phi\; e^{-i\left[\frac{\omega}{\theta_1}(\kappa + \phi)\right]} \int_0^{+\infty} dr\; r f(re^{i\phi}) \\
&\quad \frac{1}{\theta_1} \int_0^{2\pi} du\; e^{-i\left[(\frac{\omega}{\theta_1})u + kr sin u\right]} \\
\hat{\tilde{f}}(k, \kappa, \Omega; \theta_1) &= \frac{1}{\theta_1} e^{-i\Omega\,\kappa} \int_0^{+\infty} r\, dr \hat{f}(\Omega, r) J_\Omega(kr) \\
&= \frac{1}{\theta_1} e^{-i\Omega\,\kappa}\; H_\Omega[\hat{f}, k]
\end{aligned} \tag{3}$$

where $\vec{k}$ and ω are the spatial and temporal frequencies respectively. In this calculation, we have successively applied a change of variable $\vec{x}' = R(\theta_1 t)\vec{x}$, and a change from Cartesian to polar coordinates in both original and Fourier domains with $\vec{x} = (x_1, x_2) \rightarrow (r, \phi)$ taking as origin the center of rotation located at $\vec{x} = \vec{b}$ and $\vec{k} = (k_1, k_2) \rightarrow (k, \kappa)$. We let $\Omega = \frac{\omega}{\theta_1}$. The integration on variable ϕ leaves the Fourier transform $\hat{f}$ in variable Ω. The change of coordinates, the rotational transformation and the integration on time t introduces Bessel functions of real indices $J_\Omega(kr)$

$$J_\Omega(kr) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i[\Omega u + kr sin u]}\, du \tag{4}$$

The integration on the argument r leads to generalized Hankel transforms $H_\Omega[f, k]$ defined as

$$H_\Omega[f, k] = \int_0^{+\infty} f(r)\; J_\Omega(kr)\; r dr \tag{5}$$

If $f(r)$ belongs to the $L^2(\mathbf{R}, dx)$-class i.e. is square-integrable, then the Hankel transform is also L^2.

The influence of the symmetries on the temporal frequencies can be studied considering a spinning square shape. If the square is rotating with an angular velocity of $\theta = 45^o$ per image (i.e. between two consecutive image), the spectrum reaches the Shannon sampling bound beyond which the spectrum is undergoing aliasing (Figures 4 and 5). The spin of any symmetrical polygon of 2n edges will then reach the bound at $\frac{\pi}{2n}$.

3 Rotational Motion

The transformation of the rotational motion that has been stated in Equation 2 is part of the harmonic oscillator which belongs to the $SU(2)$ group [6]. Let $\theta_1 \in \mathbf{R}$ be the angular velocity, $\vec{b} \in \mathbf{R}^2$ be the spatial translation, $\tau \in \mathbf{R}$ be the temporal translation, the wavelet representation reads

$$\left[T(g)\hat{\Psi}\right](\vec{k}, \omega) = e^{i(\omega\tau + R(-\theta_1\tau)\vec{k}\cdot\vec{b})} \hat{\Psi}\left(\vec{k}, \omega\right) \tag{6}$$

These wavelet transforms are tuned to the angular velocity of interest $\theta_1\tau$. The $SU(2)$ group is compact, the representation is square integrable and then leads to a wavelet transform. The integration on $\theta_1\tau$ leads to Bessel functions. Quite general information about Bessel functions and group representations may be found in [5]. The harmonic oscillator will be now put into motion on a trajectory.

4 Translational Motion

The wavelet tuning to translational motion are naturally located in this plane, they have been described in [3], [7], [8]. A simplified version is considered here that takes into account spatial scale $a \in \mathbf{R}^+\backslash\{0\}$, spatial translation $\vec{b} \in \mathbf{R}^2$, temporal translation $\tau \in \mathbf{R}$, spatial orientation $R(\theta_0) \in SO(2)$ and velocity $\vec{v} \in \mathbf{R}^2$. The group element reads

$$g = \{\vec{b}, \tau, \vec{v}, a, R(\theta_0)\} \equiv \begin{pmatrix} a\,R(\theta_0) & \vec{v} & \vec{b} \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

The group representation in the spatio-temporal Hilbert space of the signal reads [8]

$$\left[T(g)\hat{\Psi}\right](\vec{k}, \omega) = a\; e^{i(\omega\tau + \vec{k}\cdot\vec{b})} \hat{\Psi}\left(aR(-\theta_0)\vec{k}, \omega + \vec{k}\cdot\vec{v}\right) \tag{8}$$

This representation is unitary, irreducible and square integrable [3] and generalizes to families of wavelets tuned to acceleration γ_0 as follows $[T(g)\hat{\Psi}]\left(m_2, \vec{k}, \omega, \sigma\right)$

$$= a^{n/2} e^{i(m_2\phi_2 + \vec{k}\cdot\vec{b} + \omega\tau + \sigma\tau^2)} \hat{\Psi}\left(m_2^{'}, \vec{k}^{'}, \omega^{'}, \sigma^{'}\right)$$

with $\vec{k}^{'} = aR(-\theta_0)[\vec{k} + m_2\vec{\gamma}_0]$, $\omega^{'} = \omega + \vec{k}\cdot\vec{v}_0$, $\sigma^{'} = \sigma + \vec{k}\cdot\vec{\gamma}_0 + \frac{1}{2}m_2\vec{\gamma}_0^2$, $m_2^{'} = a^2 m_2$. σ is the Fourier variable for t^2 that is considered as an independent time variable. m_2 is the parameter of the central extension [9].

5 Composing both Transformations

To derive the group that takes into account the harmonic oscillator on the carrier trajectory, rotational motion and translational motion are going to be composed. This composition provides new Lie groups with representations in $L^2(\mathbf{R}^2 \times \mathbf{R}, d^2\vec{x}dt)$ that read $[T(g)\Psi](\vec{x}, t)$

$$[T(g)\Psi](\vec{x}, t) = a^{-1}\Psi\left(\vec{x}^{'}, t^{'}\right) \tag{9}$$

with $\vec{x}^{'} = aR(-\theta_0)R(-\theta_1\tau)R(-\theta_0\tau^2)\vec{x} - \vec{v}t - \vec{\gamma}_0\frac{t^2}{2} - \vec{b}$ and $t^{'} = t - \tau$. These representations are still unitary, irreducible and square integrable if one considers position and velocities (translational and rotational) ($g_1 = \{\vec{b}, \tau, \vec{v}, \theta_0, \theta_1, a\}$) and position and accelerations (translational and rotational) ($g_1 = \{\vec{b}, \tau^2, \vec{\gamma}_0, \theta_0, \theta_2, a\}$) separately and successively. This approach leads to constructing (i.e. estimating and tracking) motion trajectories with Viterbi algorithms based on dynamic programming [10].

6 Rotational Wavelet Transform

Let the signal subject to analysis be $S(\vec{x}, t)$ and be defined in the Hilbert space $L^2(\mathbf{R}^2 \times \mathbf{R}, d^2\vec{x}dt)$. The wavelet transform $W_\Psi[S; g]$, with $g = \{\vec{b}, \tau, \vec{v}, a, \theta_0, \theta\}$ is defined as a linear mapping W_Ψ: $\Psi \to < \Psi_g | S >$ i.e. an inner product that computed in the Fourier domain and discretized on the sampling grid

$$W[s; g] = c_\Psi^{-1/2} \int_{\mathbf{R}^2\times\mathbf{R}} d^2\vec{k} d\omega \Psi_g\left(\vec{k}, \omega\right) \bar{S}\left(\vec{k}, \omega\right)$$

where the overbar $^-$ stands for the complex conjugate. The wavelet, Ψ, is a *mother wavelet.* It must satisfy the condition of admissibility calculated from square integrability [3, 8] The condition of admissibility of the accelerated wavelet is then given by $c_\Psi < \infty$ with $c_\Psi =$

$$\int_{\mathbf{R}^2\times\mathbf{R}\times\mathbf{R}} \frac{|\hat{\Psi}(\vec{k}, \omega, \sigma)|^2}{|\vec{k}|^4} \frac{\left(|\vec{k}|^2 + m_2(\sigma^{'} - \sigma)\right)}{m_2} d\vec{k}\, d\omega\, d\sigma. \tag{10}$$

7 Tracking of Rotational Motion

In this section, we develop an algorithm combining Kalman filter, rotational and translational wavelet transforms, and a dynamical structure (a Lie algebra) for the damped, harmonically driven, harmonic oscillator. According to the previous sections, the estimation and the tracking may be done globally on all the parameters or split into two parts focusing first on the trajectory and secondly, if there is interest, on the spinning motion. That kind of tracking is obtained by diagonalizing the prediction matrix. Let the state equation of the system be given by

$$\vec{u}(t) = \{x(t), v(t), \cos[\theta_1 t], \sin[\theta_1 t]\}^T \tag{11}$$

The evolution is given by the prediction equation

$$\vec{u}(t+\tau) = e^{t\Omega}\vec{u}(t) = e^{tA^T}\vec{u}(t) \tag{12}$$

where Ω is an operator of the Lie algebra (a sub-algebra of $gl(4, \mathbf{R})$) and A is a 4×4 matrix obtained from the operator Ω. It can be formulated as

$$A = \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ 1 & -\beta & 0 & 0 \\ 0 & \gamma & 0 & \theta_1 \\ 0 & 0 & -\theta_1 & 0 \end{pmatrix} \tag{13}$$

we have also

$$\vec{u}(t+\tau) = \{x(t+\tau), v(t+\tau), \cos[\theta_1(t+\tau)], \sin[\theta_1(t+\tau)]\}^T \tag{14}$$

To yield an interesting closed form of the prediction equation that splits translational motion from rotational motion, it is desirable to diagonalize the matrix e^{tA^T}. Eventually, the calculations lead to two 2-component relations as follows

$$\begin{pmatrix} x(t+\tau) \\ v(t+\tau) \end{pmatrix} = V^{-1}D_1V\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} + \left[\left(V^{-1}D_1V\right)C - C\left(U^{-1}\, D_2\, U\right)\right]\begin{pmatrix} \cos(\theta_1 t) \\ \sin(\theta_1 t) \end{pmatrix} \tag{15}$$

and

$$\begin{pmatrix} \cos[\theta_1(t+\tau)] \\ \sin[\theta_1(t+\tau)] \end{pmatrix} = \left(U^{-1}D_2U\right)\begin{pmatrix} \cos(\theta_1 t) \\ \sin(\theta_1 t) \end{pmatrix} \tag{16}$$

The tracking strategy is based on combining *Kalman filters* and *wavelet transform.* The state of the Kalman filter is currently composed of some 11 or all the wavelet parameters ([8] for translational case).

The observation equation also exploits the wavelet transform as a motion-based extraction tool tuned to the current exact state parameters. The CWT captures and isolates the selected objects from the scene S to provide a display I,

$$I(\vec{b}, \tau) = < \Psi_{g_{opt}} | S + V > . \tag{17}$$

I is the segmented image of the selected object, displayed alone at its correct location; S is the original signal under analysis, $V(\vec{b}, \tau)$ is the noise produced by the optical sensors and g_{opt} is the set of optimal parameters corresponding to the estimation of state parameters which is performed with Morlet wavelets as described in the next section.

8 Morlet wavelet and applications

The applications presented in this paper are based on Morlet wavelets. An anisotropic *Morlet wavelet* is admissible as an continuous wavelet in the rotational and translational family. The still 3-D+T Morlet wavelet defines a non-separable filter

$$\Psi(\vec{x}, t) = e^{i\vec{k}_0\vec{X}}\ e^{-\frac{1}{2}<\vec{X}\ |\ C\vec{X}>} - e^{-\frac{1}{2}<\vec{k}_0\ |\ D\vec{k}_0>}\ e^{-\frac{1}{2}<\vec{X}|C\vec{X}>} \tag{18}$$

where $\vec{X} = (\vec{x}, t)^T \in \mathbf{R}^2 \times \mathbf{R}$, C is a positive definite matrix and, $D = C^{-1}$. For $2D + T$ signals, $\left[C = \begin{pmatrix} 1/\epsilon_x & 0 & 0 \\ 0 & 1/\epsilon_y & 0 \\ 0 & 0 & 1/\epsilon_t \end{pmatrix} \right]$ where the ϵ factors introduce anisotropy in the wavelet shape. Figures 4 and 5 presents three configurations of rotational wavelets: four symmetrical 2-D spatial Morlet wavelets are considered in the space and time domain, and successively put in rotational motion, put into velocity and transformed in the Fourier domain where the inner product with signal is taking place. Figure 6 presents the detection of scale and angular velocity using the square of the modulus of the rotational Morlet wavelet. This estimation is performed by integrating over the whole image sequence the square modulus of wavelet transform at velocity $\vec{v}_0 = (-2.7, -0.1)$ to determine

$$\max \arg_{\theta,a} \int_{\mathbf{R}^2} \int_{\mathbf{R}} d\vec{b} d\tau \ \ | < \Psi_{\vec{v}=\vec{v}_0, \theta t, a, \vec{b}, \tau} | S > |^2 \tag{19}$$

9 Conclusions

This paper has presented a new practical and efficient way of estimating and tracking trajectories with new spatio-temporal wavelets. The technique is robust against image noise, motion jitter and temporary occlusions [3], [7], [8]. The motion analysis performed by the wavelet transform may be split into two distinct parts: the first estimates locations and velocities and tracks the trajectory, the second estimates the angular velocity. The object in motion is clearly subject to two distinct motions, a displacement and a spinning. This analysis is supported by the Fourier signatures of these motions. Some further theoretical and applied works are dealing with continuous wavelet transforms to analyze motion on manifolds. Indeed, the Lie group and algebra theory allow the derivation of square integrable representations of kinematical groups for motion on manifolds. In this study, the Lie generators for translational, rotational and deformational motion are merging harmoniously to new Lie generators of motion for example on homogeneous spaces like spheres, paraboloids and hyperboloids. At that point, Lie group representations, mechanics on manifolds, differential geometry, Kalman filtering, optimum control, continuous wavelets and symmetry properties all merge and generalize into a unique theoretical framework. The work developed in this paper has been a seed towards this theoretical achievement [10].

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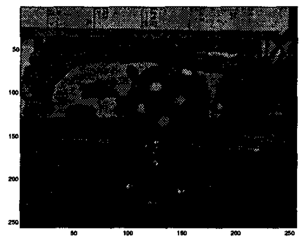

Figure 1: Image 15 in the Caltrain sequence ([256 × 256] × 32 images): the purpose is to analyze the ball which is spinning (four white spots), and moving on a quasi-horizontal trajectory with a deceleration from images 1 to 14 followed by an acceleration from images 15 to 32.

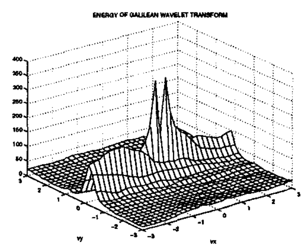

Figure 2: Analysis of the velocities contained in the ball at image 18. The ball is windowed and the square modulus of the Galilean wavelet is integrated on the ball domain at image 18: two signatures are visible, the two symmetric peaks for the rotational motion and a "domed wall" for the accelerated motion.

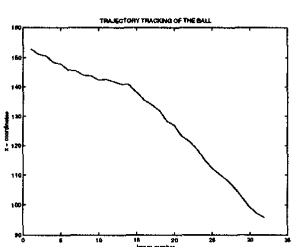

Figure 3: Tracking the carrier trajectory of the moving ball position in the Caltrain sequence with Galilean wavelets. At image 15, the ball is pushed and keeps constant speed. Coordinates decreases as a result of the motion steering to the left.

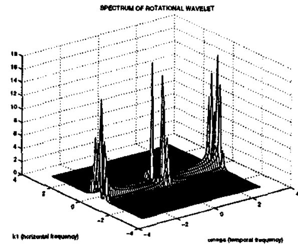

Figure 4: Four symmetrical rotational Morlet wavelet functions close to $\theta t = (\pi/4)t$, and $\vec{v} = \{0,0\}$: square modulus in plane of the (ω, k_x) axes i.e. $k_y = 0$. This yields critical temporal sampling.

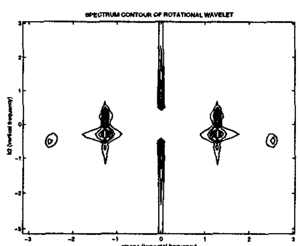

Figure 5: Four symmetrical rotational Morlet wavelet function at $\theta t = (\pi/10)t$, and $\vec{v} = \{0,0\}$: contours of the square modulus in plane of the (ω, k_y) axes i.e. $k_y = 0$. This yields $4 \times \pi/10$ on the temporal frequency axis.

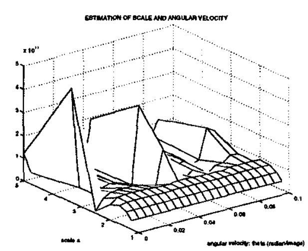

Figure 6: Estimation with rotational wavelets of the angular velocity and the scale of the rotating ball in caltrain sequence. The diagram sketches the square of the modulus of the wavelet transform at $\vec{v} = \{-2.7, 0.1\}$. The component at $\theta = 0$ stands for the non rotating structures. The component at $\theta = 0.045, a = 3.3$ is the actual ball contribution which is observed rotating of $\pi/2$ over 32 images. $\theta = 0.09$ is a harmonic.