Numerical Harmonic Analysis

The purpose of this talk is to describe recent developments involving the numerical implementation of methods from classical harmonic analysis in signal processing and computational P.D.E.

As an example, Littlewood-Paley theory, in which a function or a Fourier multiplier is analyzed by partitioning the frequency space in dyadic blocks, has recently been translated into a powerful numerical tool through expansions in orthonormal wavelet bases. (See [1],[2].)

In this numerical setting one sees a general Calderon-Zygmund operator or ΨD.O. as given by an "almost" diagonal matrix having a simple analysis and being implementable by fast numerical algorithms (i.e. algorithms of complexity $CN\log\ N$, $N=$ number of discretization points:) Pseudo-differential calculus is translated into an efficient numerical calculus in which smoothing operators are represented by "small" matrices of low numerical rank (see [1]) permitting its use in explicit calculations of solutions to P.D.E. In particular, we can obtain a fast algorithm for the numerical computation of the Green's function for a variable coefficient Laplacean (with smooth coefficients).

In this exercise of translation of methods and ideas from harmonic analysis into fast computational algorithms, one soon realizes that the ability to implement efficiently an integral operator applied to a function is equivalent to a good understanding of the interaction between geometry of the underlying space and cancellation properties of the operator. In the particular case of Calderon-Zygmund operators we see an efficient m-computational algorithm as being a translation of the method of proof of the T of 1 theorem of David and Journé. For the case of fractional integrals and operators of potential theory, the need to come up with efficient computations has led V. Rokhlin to the independent discovery of various versions of Calderon-Zygmund theory as embodied in his multipole algorithms.

As it turns out, in this case, the question of fast computation is more elementary than boundedness on L^2 or other spaces. It leads directly to issues of geometry of interactions and cancellations.

This interaction between harmonic analysis and a number of concrete problems in applications, such as signal processing and computations, has opened a number of new fundamental questions in analysis.

Our goal is to describe some of these problems on a few simple examples. We start with a fundamental question of signal processing, the question of compression of a signal. Stated simply, given a function (or more precisely, a vector which is a sampled function) one would like to represent the function with as few parameters as possible (here a representation is always assumed to a given fixed precision). Such a representation could be given in terms of expansion coefficients, Fourier, Taylor, etc. or by stating that the function solves an equation which is easy to describe (say by giving coefficients of a differential equation). The ability to represent a function simply with few parameters is not only desirable in applications for storage purposes, it is also a test of our understanding of the structure of the function and its numerical complexity. Traditionally, the first attempt to represent a signal (or a function not described analytically) would be to expand the signal in a Fourier seires, or in terms of some other orthogonal (or non orthogonal expansion). This leads to a variety of problems familiar to all analysts. Assume that a smooth function is supported on a number of disjoint intervals. It is "clear" that separate Fourier expansions restricted to these intervals will be much more "efficient" than a single expansion on the union. The actual answers are not so obvious since some intervals could be close to each and the term efficient has not been defined. We see that we are confronted with the issue of selecting an optimal expansion inside a class of possible expansions. This leads naturally to the concept of a library of orthonormal bases, as well as to precise definitions of efficiency of an expansion.

Definition of modulated wave form libraries

We start by observing that it is impossible to construct an orthogonal basis by localizing smoothly e^{ikx}. This is clear for the case of two adjacent windows $w_1(x), w_2(x)$ since the requirement of orthogonality between $w_1 e^{ikx}$ and $w_2(x)e^{ijx}$

implies that

$$\int w_1(x)w_2(x)e^{i(k-j)x}dx = 0$$

which implies $w_1(x)w_2(x) \equiv 0$ (if it is supported in an interval of length smaller than 2π).

Recently Daubechies, Jaffard, and Journé, as well as Malvar, observed that by taking equal windows and sines or cosines orthogonality can be maintained. It was observed in [3] that the windows can be chosen to different sizes enabling adaptive constructions . (See Figures 5,6)

Local trigonometric waveforms

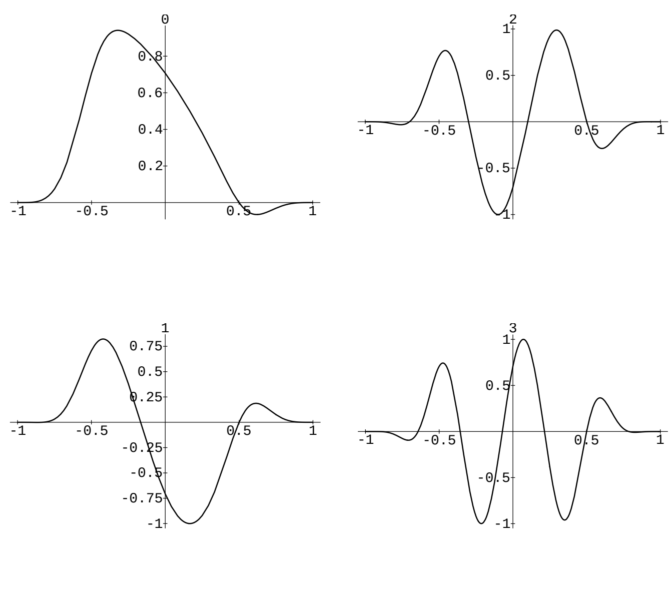


Figure 1

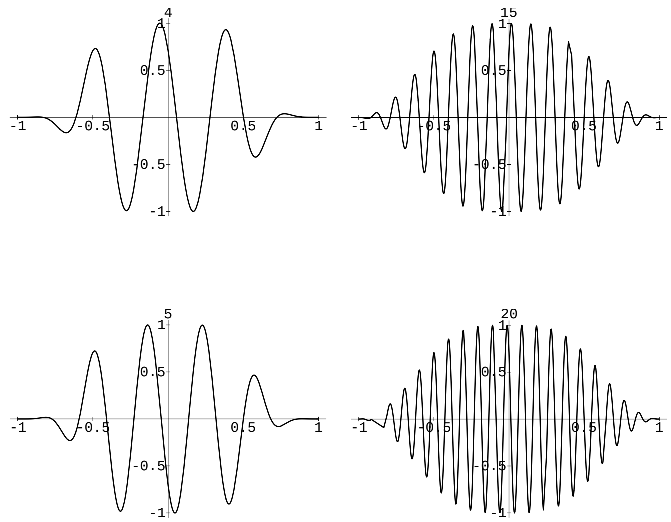


Figure 2

We start by defining this library of trigonometric waveforms. These are localized sine transforms associated to covering by intervals of $\mathbf{R}$ (more generally, of a manifold).

We consider a cover $\mathbf{R} = \bigcup_{-\infty}^{\infty} I_i \; I = [\alpha_i \alpha_{i+1}) \; \alpha_i < \alpha_{i+1}$, write $\ell_i = \alpha_{i+1} - \alpha_i = |I_i|$ and let $p_i(x)$ be a window function supported in $[\alpha_i - \ell_{i-1}/2, \alpha_{i+1} + \ell_{i+1}/2]$ such that

$$\sum_{-\infty}^{\infty} p_i^2(x) = 1$$

and

$$p_i^2(x) = 1 - p_i^2(2\alpha_{i+1} - x) \quad \text{for} \quad x \quad \text{near} \quad \alpha_{i+1}$$

then the functions

$$S_{i,k}(x) = \frac{2}{\sqrt{2\ell i}} p_i(x) \sin[(2k+1)\frac{\pi}{2\ell_i}(x - \alpha_i)]$$

form an orghonormal basis of $L^2(\mathbf{R})$ subordinate to the partition p_i. The collection of such bases forms a library of orthonormal bases.

It is easy to check that if H_{I_i} denotes the space of functions spanned by $S_{i,k}$ $k = 0, 1, 2, \ldots$ then $H_{I_i} + H_{I_{i+1}}$ is spanned by the functions

$$P(x) \frac{2}{\sqrt{2(\ell_i + \ell_{i+1})}} \sin[(2k+1)\frac{\pi}{2(\ell_i + \ell_{i+1})}(x - \alpha_i)]$$

where

$$P^2 = p_i^2(x) + p_{i+1}^2(x)$$

is a "window" function covering the interval $I_i \cup I_{i+1}$. This fundamental identity permits the useful implementation of the adapted window algorithm described in Figure 1. (Other possible libraries can be constructed. The space of frequencies can be decomposed into pairs of symmetric windows around the origin ,on which a smooth partition of unity is constructed.

Higher dimensional libraries can also be easily constructed,(as well as libraries on manifolds) leading to new and direct analysis methods for linear transformations.)

Relation to Wavelets - Wavelet Packets.

We consider the frequency line $\mathbf{R}$ split as $\mathbf{R}^+ = (0, \infty)$ union $\mathbf{R}^- = (-\infty, 0)$. On $L^2(0, \infty)$ we introduce a window function $p(\xi)$ such that $\sum_{k=-\infty}^{\infty} p^2(2^{-k}\xi) = 1$ and $p(\xi)$ is supported in $(3/4, 3)$ clearly we can view $p(2^{-k}\xi)$ as a window function above the interval $(2^k, 2^{k+1})$ and observe that

$$\sin\left[(j + \frac{1}{2})\pi\left(\frac{\xi - 2^k}{2^k}\right)\right] p(2^{-k}\xi) = s_{k,j}$$

form an orthonormal basis of $L^2(\mathbf{R}^+)$. Similarly $c_{k,j} = \cos\left[(j + \frac{1}{2})\pi\left(\frac{\xi - 2^k}{2^k}\right)\right] p(2^{-k}\xi)$ gives another basis. If we define $S_{k,j}$ as an odd extension to $\mathbf{R}$ of $s_{k,j}$ and $C_{k,j}$ as an even extension, we find $S_{k,j} \perp C_{k',j'}$ permitting us to write $C_{k,j} \pm iS_{k,j} = e^{\pm ij\pi\xi/2^k}\hat{\psi}(\xi/2^j)$ where $\hat{\psi}(\xi) = e^{i\pi/2\xi}p(\xi)$ is the Fourier transform of the base wavelet Ψ (see Meyer).

We therefore see that wavelet analysis corresponds to windowing frequency space in "octave" windows $(2^k, 2^{k+1})$.

A natural extension therefore is provided by allowing all dyadic windows in frequency space and adapted window choice. This sort of analysis is "equivalent" to wavelet packet analysis.

The wavelet packet analysis algorithms permit us to perform an adapted Fourier windowing directly in time domain by successive filtering of a function into different regions in frequency. The dual version of the window selection provides an adapted subband coding algorithm.

This new library of orthonormal bases constructed in time domain is called the Wavelet packet library. This library contains the wavelet basis, Walsh functions, and smooth versions of Walsh functions called wavelet packets.See Figure 7

Wavelet Packet Library

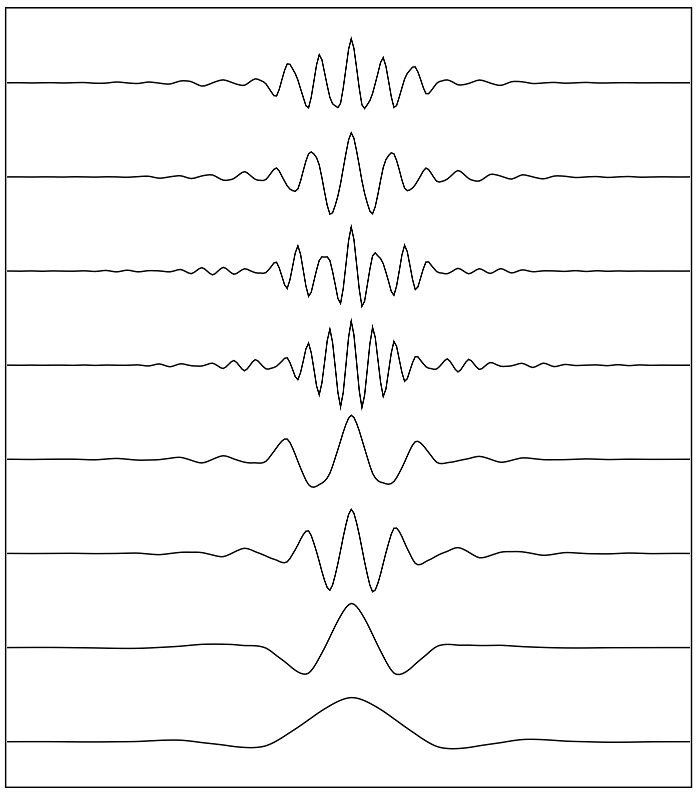


Figure 3

We'll use the notation and terminology of [4], whose results we shall assume.

We are given an exact quadrature mirror filter $h(n)$ satisfying the conditions of Theorem (3.6) in [4], p. 964, i.e.

$$\sum_n h(n-2k)h(n-2\ell) = \delta_{k,\ell}, \ \sum_n h(n) = \sqrt{2}.$$

We let $g_k = h_{l-k}(-1)^k$ and define the operations F_i on $\ell^2(\mathbf{Z})$ into "$\ell^2(2\mathbf{Z})$"

$$(1.0) \qquad F_0\{s_k\}(i) = 2\sum s_k h_{k-2i}$$

$$F_1\{s_k\}(i) = 2\sum s_k g_{k-2i}$$

The map $\mathbf{F}(s_k) = F_0(s_k) \oplus F_1(s_k) \in \ell^2(2\mathbf{Z}) \oplus \ell^2(2\mathbf{Z})$ is orthogonal and

$$F_0^* F_0 + F_1^* F_1 = I \tag{1.1}$$

We now define the following sequence of functions.

$$\begin{cases} W_{2n}(x) = \sqrt{2} \sum h_k W_n(2x-k) \\ W_{2n+1}(x) = \sqrt{2} \sum g_k W_n(2x-k). \end{cases} \tag{1.2}$$

Clearly the function $W_0(x)$ can be identified with the scaling function φ in [D] and W_1 with the basic wavelet ψ.

Let us define $m_0(\xi) = \frac{1}{\sqrt{2}} \sum h_k e^{-ik\xi}$ and

$$m_1(\xi) = -e^{i\xi} \bar{m}_0(\xi + \pi) = \frac{1}{\sqrt{2}} \sum g_k e^{ik\xi}$$

Remark. The quadrature mirror condition on the operation $\mathbf{F} = (F_0, F_1)$ is equivalent to the unitarity of the matrix

$$\mathcal{M} = \begin{bmatrix} m_0(\xi) & m_1(\xi) \\ m_0(\xi+\pi) & m_1(\xi+\pi) \end{bmatrix}$$

Taking the Fourier transform of (1.2) when $n = 0$ we get

$$\hat{W}_0(\xi) = m_0(\xi/2)\hat{W}_0(\xi/2)$$

i.e.,

$$\hat{W}_0(\xi) = \prod_{j=1}^{\infty} m_0(\xi/2^j)$$

and

$$\hat{W}_1(\xi) = m_1(\xi/2)\hat{W}_0(\xi/2) = m_1(\xi/2) m_0(\xi/4) m_0(\xi/2^3) \cdots$$

More generally, the relations (1.2) are equivalent to

$$\hat{W}_n(\xi) = \prod_{j=1}^{\infty} m_{\varepsilon_j}(\xi/2j) \tag{1.3}$$

and $n = \sum_{j=1}^{\infty} \varepsilon_j 2^{j-1} (\varepsilon_j = 0 \text{ or } 1)$.

The functions $W_n(x-k)$ form an orthonormal basis of $L^2(\mathbf{R}^1)$. a We define a library of wavelet packets to be the collection of functions of the form $W_n(2^\ell x - k)$

where $\ell, k \in \mathbf{Z}, n \in N$. Here, each element of the library is determined by a scaling parameter ℓ, a localization parameter k and an oscillation parameter n. (The function $W_n(2^\ell x - k)$ is roughly centered at $2^{-\ell}k$, has support of size $\approx 2^{-\ell}$ and oscillates $\approx n$ times).

We have the following simple characterization of subsets forming orthonormal bases.

Proposition. *Any collection of indices (ℓ, n) such that the intervals $[2^\ell n, 2^\ell n + 1)$ form a disjoint cover of $[0, \infty)$ gives rise to an orthonormal basis of L^2*[1].

(These intervals correspond to the partition of frequency space alluded to in §1.)

Motivated by ideas from signal processing and communication theory we were led to measure the "distance" between a basis and a function in terms of the Shannon entropy of the expansion. More generally, let H be a Hilbert space.

Let $v \in H,\ \|v\| = 1$ and assume

$$H = \oplus \sum H_i$$

an orthogonal direct sum. We define

$$\varepsilon^2(v, \{H_i\}) = -\sum \|v_i\|^2 \ell n \|v_i\|^2$$

as a measure of distance between v and the orthogonal decomposition.

ε^2 is characterized by the Shannon equation which is a version of Pythagoras' theorem.

Let

$$\begin{aligned} H &= \oplus(\sum H^i) \oplus (\sum H_j) \\ &= H_+ \oplus H_- \end{aligned}$$

H^i and H_j give orthogonal decompositions $H_+ = \sum H^i, H_- = \sum H_j$. Then

$$\begin{aligned} \varepsilon^2(v; \{H^i, H_j\}) = \varepsilon^2&(v, \{H+, H_-\} \\ &+ \|v_+\|^2 \varepsilon^2 \left(\frac{v_+}{\|v_+\|}, \{H^i\} \right) \\ &+ \|v_-\|^2 \varepsilon^2 \left(\frac{v_-}{\|v_-\|}, \{H_j\} \right) \end{aligned}$$

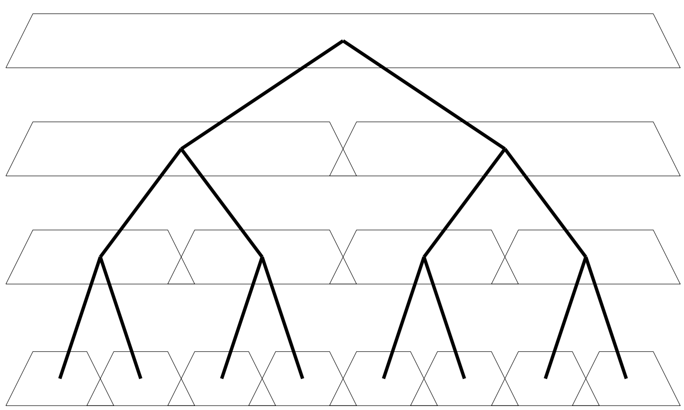
[1] We can think of this cover as an even covering of frequency space by windows roughly localized

This is Shannon's equation for entropy (if we interpret as in quantum mechanics $\|P_{H_+}v\|^2$ as the "probability" of v to be in the subspace H_+).

This equation enables us to search for a smallest entropy space decomposition of a given vector.

In fact, for the example of the first library restricted to covering by dyadic intervals we can start by calculating the entropy of an expansion relative to a local trigonometric basis for intervals of length one, then compare the entropy of an adjacent pair of intervals to the entropy of an expansion on their union. Pick the expansion of minimal entropy and continue until a minimum entropy expansion is achieved (see Figure 4).

Schematic Description



Of course, while entropy is a good measure of concentration or efficiency of an expansion, various other information cost functions are possible, permitting discrimination and choice between various expansions.

We illustrate these points, as well as the effect of various analysis methods, in the next figures in which the vertical axes represents the frequency axes and the horizontal is the time (or space) axes. The signals have 512 samples (and are wrapped around). Each rectangular box in this phase space corresponds to a coefficient obtained by correlating the signal with an element of the wavelet packet library whose time support lies "below" the box and whose frequency support is in the projection of the box on the vertical axis. Each box has area 512 pixels (i.e. a cover of the discrete phase plane has 512 element).

The compression rate can be computed as the ratio of the visible gray area to the total area of the box (i.e. the relative number of visible boxes).

Wavelet coefficients of a function.

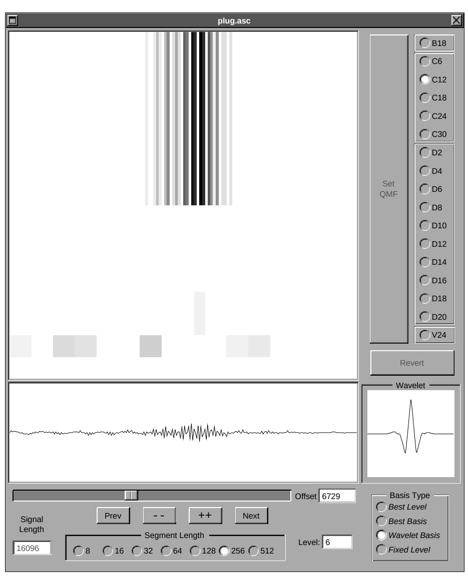


Figure 4

A bestbasis wavelet packet analysis, this analysis corresponds

to selection of windows in frequency space,

to minimize the entropy of the expansion.

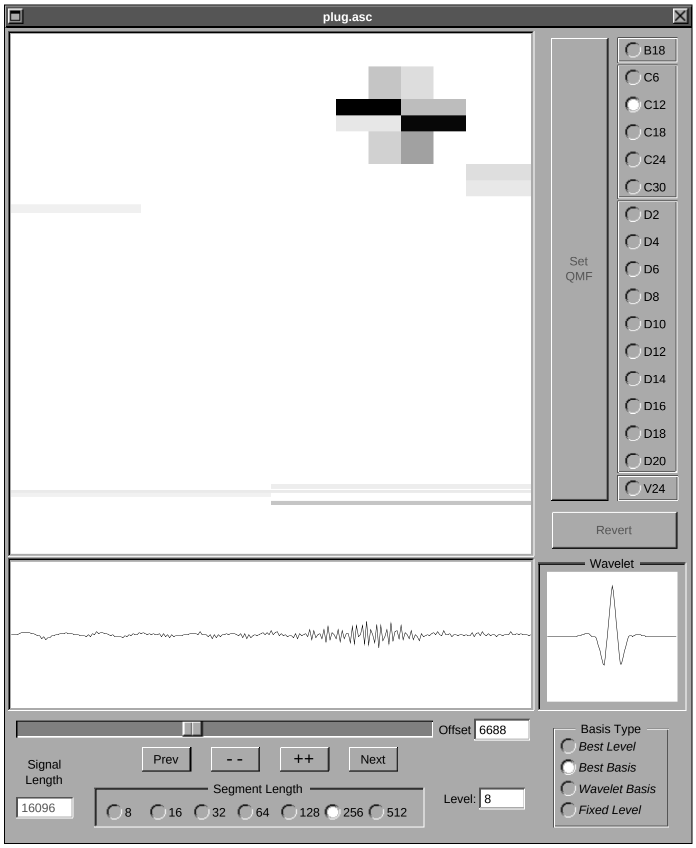


Figure 5

A two windows expansion,with no adaptation

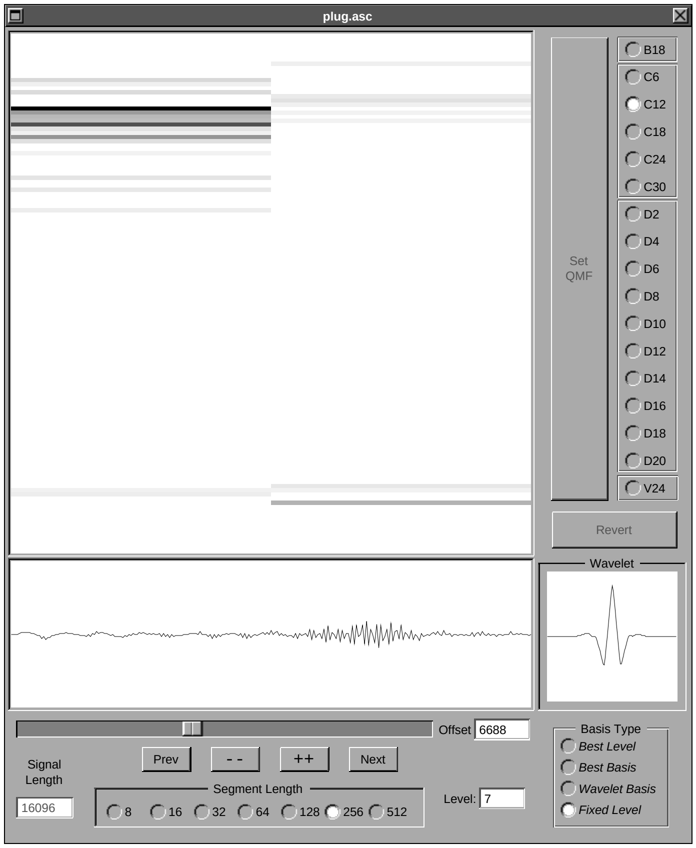


Figure 6

A best level expansion,in which a fixed window size is chosen to minimize entropy

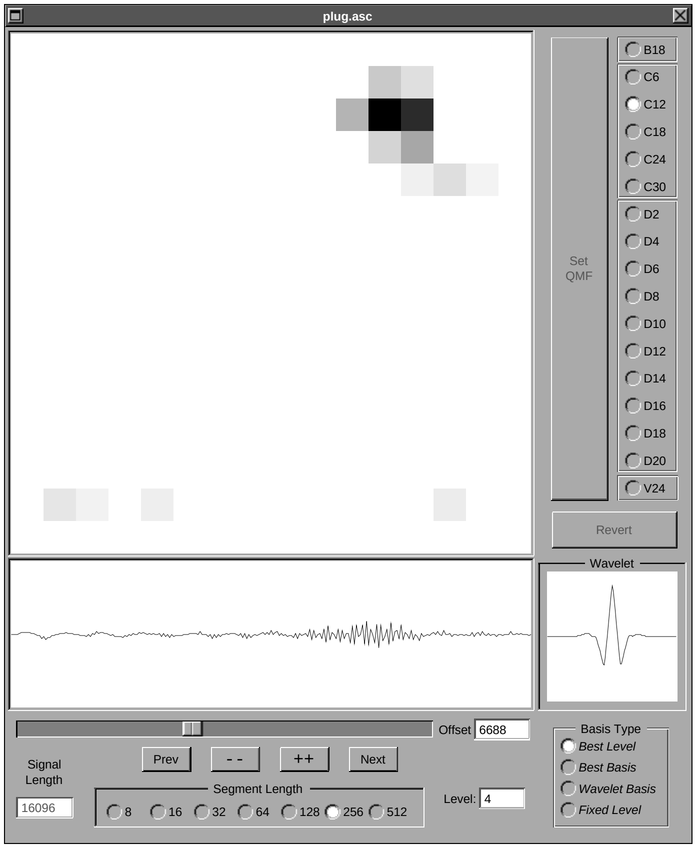


Figure 7

Of course we can try to characterize classes of functions which are well compressible i.e., for which we can estimate the number of coefficients needed for representing the function with a prescribed accuracy. Smooth functions are obvious candidates as well as functions which can be well approximated locally by trigonometric polynomials of short length. Various obvious definitions come to mine. At the moment it would seem that experiment will provide a better guide.

The procedure for signal analysis described above is very similar to the usual methods of studying Fourier multipliers in which we break the multiplier by an appropriate partition of unity to simpler components whose spatial localization and structure are easier to understand. We can describe a similar procedure for integral operators

$$T(f) = \int k(x, y) f(y) dy$$

which are not necessarily convolutions.

Our goal is to implement a discrete version of the operator fast.

A procedure that is equivalent to P_tQ_t decompositions of Calderon-Zygmund operators (see []) can be obtained by trying to compress the $k(x,y)$ viewed as an image, i.e. $k(x,y)$ represents light intensity at pixel (x,y). Here again the analysis consists in finding an optimal windowed expansion for $k(x,y)$ (or $\hat{k}(\xi,\eta)$) by selecting that combination of windows most efficient in capturing the kernel (see Figure).

Since the kernel is represented as a sum of products of functions of x and y it is easy to convert an efficient two dimensional representation into a corresponding efficient computation. Observe also that each box selected represents an interaction between two windows on the line.

It can be proved that for $k(x,y)$ a single or double layer potential for Helmholtz on a curve or surface (with bounded curvature). This procedure leads to an order $pN \log\ N$ algorithm, where p is the number of decimals desired, N is the number of discretization points ($\approx$ number of wave lengths on the surface).

For more general curves or surfaces one has to develop specific, highly oscillatory, analogues to multipoles. (Smooth bases are useless). This has been done by Rokhlin with a resulting description of local oscillatory interactions.

References

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6. Available by anonymous ftp, *ceres.math.yale.edu*, InterNet address 130.132.23.22.