REVIEWS

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Ripples in Mathematics: The Discrete Wavelet Transform. By Arne Jensen and Anders la Cour-Harbo. Springer, Berlin, 2001, ix + 246 pp., ISBN 3-540-41662-5, $39.95.
A First Course in Wavelets with Fourier Analysis. By Albert Boggess and Francis J. Narcowich. Prentice Hall, Upper Saddle River, NJ, 2001, 260 pp., ISBN 0-13-022809-5, $50.00.

Reviewed by **M. Victor Wickerhauser**

1. THE FRUSTRATING PROBLEM. Fourier analysis with sines and cosines has been a staple of pure and applied mathematics research since 1822. It has spread to quantitative sciences and engineering, trickling down throughout their college curricula. Nowadays a student may be introduced to orthogonal Fourier series shortly after learning to integrate sine and cosine. Without much more than integration by parts, the relationship between smoothness of a periodic function and decrease of its Fourier coefficients, and even a simple theorem on convergence of Fourier series, can be proved rigorously. At this point, the student studying this material in order to be a better engineer will want to apply Fourier analysis to the nonsmooth or nonperiodic or sampled functions encountered in practice. Wanting equal rigor, that student will need to take advanced calculus to understand where and how the Fourier series of a periodic step function converges, and then go on to measure theory and functional analysis to understand the Fourier integral transform on nonperiodic functions in L^2 or tempered distributions such as Dirac's delta. This represents most of a mathematics doctoral candidate's analysis preparation; it is too much to expect of an undergraduate non-major.

It is frustrating for us mathematicians to leave students with a limited understanding of such an important tool. The failure is exacerbated by the wide use of discrete Fourier transforms for machine computation and digital signal processing, and courses for engineers and scientists have arisen as a partial remedy. Some of these are phenomenological; they compute the Fourier transform of a few examples not covered by calculus or discuss how convergence problems ("aliasing" or "artifacts" or "ringing") are solved by regularization ("filtering" or "smoothing" or "windowing"). Others restate the theorems of Fourier analysis for the discrete Fourier transform, which yields propositions in finite-dimensional linear algebra that they can prove with complete rigor. Still others focus on numerical algorithms and implementations such as the very elegant factored discrete Fourier transform on 2^n points, expecting the student to develop understanding through hands-on laboratory experience.

Wavelet analysis has been called that only since the 1980s, but it too has found its way into the undergraduate curriculum. For a while, it even looked like a better subject for early introduction, as the discrete Haar wavelet transform is arguably easier to present than the discrete Fourier transform. Just as in Fourier analysis, some of the basic results are elementary. The orthogonality and completeness of Haar functions, necessary and sufficient conditions for two sequences to be a perfect reconstruction filter pair, and even a smoothness estimate for Daubechies's wavelets can be proved rigorously using calculus and linear algebra alone. Programming a discrete wavelet transform is a good way to learn a new computer language, and filtering some signals in wavelet coordinates is a nice laboratory project. However, again as in Fourier analysis, a rigorous treatment of filters, multiresolutions, and their underlying functions needs some real and functional analysis.

In both cases, the steep mathematical prerequisites rise up like the Blue Mountains near Sydney, keeping the coastal newcomers from reaching the great continent that lies beyond. Can anyone find a path for them to take? The two books under review offer themselves as guides.

2. TWO ROUTES TO WAVELET ANALYSIS. One of the books, *A First Course in Wavelets with Fourier Analysis*, by Albert Boggess and Francis J. Narcowich, takes the mathematician's classical direct approach, though with something less than classical rigor. That is probably because the text is supposed to be "manageable for a one-semester undergraduate course" (p. xvi). Both authors are distinguished mathematicians who do research in function theory and approximation theory, respectively. The text is based on lecture notes for a course they gave at Texas A&M University.

The text starts at Chapter 0, which introduces infinite-dimensional inner-product spaces with a focus on L^2 and ℓ^2. The next three chapters are on Fourier series, the Fourier integral transform, and the discrete Fourier transform, respectively. This is the path our ancestors took, except that they were forced to stop and prove everything. The newer ground of wavelet analysis, in the four-chapter second half, is explored by a somewhat nonhistorical route. Chapter 4 is a nice development of Haar's 1910 basis for L^2, brought up to date with the notion of piecewise constant multiresolution approximation spaces, or MRAs. Chapter 5 defines MRAs in general, and defines the discrete wavelet transform. Chapter 6 is a short treatise on the useful special case of Daubechies's wavelets, with some attention paid to computational issues. At this point, the recommended material for one semester has been covered. The authors have arrived at the goal set out in the first page of the preface, the "essential ideas behind Fourier analysis and wavelets," but Chapter 7 provides some further directions, introducing wavelet packets, multidimensional wavelets, and the continuous wavelet transform.

I was struck by the "heuristic manner" (p. xvi) of proof and definition used to accelerate the presentation. Chapter 0, for example, starts with $\mathbf{R}^3$ and $\mathbf{C}^2$, and claims (p. 3) that any 2×2 nonsingular Hermitian matrix A defines a positive inner product $\langle v, w \rangle = w^* A v$, which is of course false unless A is positive definite as well. It then moves to infinite-dimensional examples with little discussion of the deep and crucial mathematical ideas needed in that case. L^2 and ℓ^2 are defined informally, without the Lebesgue integral, but the notion "set of measure zero" is mentioned with a very confusing definition on page 6, right after the norm on L^2 is defined. Equivalence of Riemann-integrable functions differing at finitely many points is introduced to make the L^2 norm nondegenerate, though of course it does not. Everyone knows how tough it is to get these things right without bogging down the course almost before it starts, but I have seen it done better elsewhere.

In other cases, a classical theorem is stated with its usual hypotheses, but proved only with much stronger assumptions. For example, Chapter 1 has a statement of the Riemann-Lebesgue lemma (Theorem 1.21, p. 61) for piecewise continuous functions, with an assertion that this hypothesis "can be weakened considerably," but the calculus-based proof that follows after some graphical motivation is valid only for differentiable functions. Shortly thereafter, convergence of the Fourier series for a continuous function at a point of differentiability (Theorem 1.22, p. 63, in the section entitled

"Convergence at a Point of Continuity") is proved by the traditional method of estimating Dirichlet's kernel, but the final step requires the Riemann-Lebesgue lemma for a continuous function.

This hypothesis switching happens even in the appendices, where "rigorous proof" (p. 261) is offered for some theorems from the main text. For example, the Fourier inversion theorem (Theorem 2.1, p. 93) is stated for f belonging to $C^1 \cap L^1$ but proved in Appendix A only for compactly supported f, and the proof quotes the Riemann-Lebesgue lemma for a piecewise continuous function, which is never rigorously proved.

In contrast, Chapter 4 on Haar analysis offers more reliable footing by sticking to piecewise constant functions. Here, besides many informative figures, there are simple, concrete formulas and mostly complete proofs. Only the proof that L^2 decomposes into an infinite direct sum of Haar wavelet subspaces (Theorem 4.9, p. 165), acknowledged to be out of reach, is omitted. The copy-editing for this part could be improved, though. Something that really sticks out—in the table of contents, no less—is that Sections 4.2.2 and 4.2.3 are both entitled "Basic Properties of the Haar Scaling Function." Another silly error is the definition of the scale space V_j on page 161 as the span of four functions, because someone left out the dots. This is aggravated by the use of four-step functions in the immediately preceding Examples 4.2 and 4.3. However, I think most students would not be confused by such bugs.

Chapter 5 also manages to avoid trouble by following Mallat's development of MRAs [**6**], which hides most of the difficult analysis in the axioms. Given these axioms (Definition 5.1, p. 184), only the details of constructing a mother wavelet (Theorem 5.10, p. 190) must be left to the appendix. The authors give Mallat's sufficient condition for a filter to generate a scaling function for an MRA by iteration (Theorem 5.23, p. 217), putting most of the proof in Appendix A. To convince the skeptical but lazy reader, the first four iterates approximating the famous Daubechies 4 scaling function are plotted. This provides a nice introduction to Chapter 6 on Daubechies's wavelets, which in turn leads naturally to the topical Chapter 7. These later chapters emphasize computation, but since the common language in the marriage of Fourier and wavelet analysis is convolutions, the more efficient lifting method of implementation [**10**] is not discussed.

Boggess and Narcowich offer the reader copious examples, figures, exercises, and code fragments for MATLAB in the text to aid the student's heuristic understanding and to make up for skirting the deeper mathematical issues. Chapter 4 on Haar wavelets has twenty-three figures in twenty-seven pages. Chapters 0–6 have 131 exercises among them. In Chapter 3, on the discrete Fourier transform $\mathcal{F}$, these range from graphing a given complex-valued function to proving the convolution theorem $\mathcal{F}[y * z]_k = \mathcal{F}[y]_k \mathcal{F}[z]_k$ and writing a MATLAB script to solve $u'' + 2u' + 2u = 3\cos(6t)$ on 256 points. Chapter 7 and the appendices have no exercises, but they are not listed as part of the course this book supports. Appendix B has the MATLAB codes needed for the exercises, both with and without calls to the extra-cost Wavelet Toolbox.

The second book under review, *Ripples in Mathematics: The Discrete Wavelet Transform*, by Arne Jensen and Anders la Cour-Harbo, could hardly be more different. It floats over the mathematical obstacles by keeping everything finite, dropping the heavy machinery needed for infinite series and function spaces. The text grew out of lecture notes for a course for Danish Engineering College teachers and was also tested on working engineers and mathematics students at Aalborg University. The authors' stated goals are to define and interpret the discrete wavelet transform and to explain the lifting implementation for computation with sampled signals. The audience is mathematics undergraduates, engineering students at all levels, and working engineers who have attained or retained a "modest" mathematical preparation. Jensen is a renowned mathematical physicist, and la Cour-Harbo is an energetic young control engineer.

Virtually everything about the discrete wavelet transform can be illustrated with eight-point signals and the Haar wavelet filters as examples, and Jensen and la Cour-Harbo take great advantage of this. The reader does not need to use a computer to reproduce most of the examples in the text, even though they are computational. Teaching from this book would proceed smoothly even with totally computer-naïve students until quite late in the course (Chapter 11 of fourteen, "Implementation"), where longer signals are used to illustrate time-frequency content and lossy compression.

Fourier series do not appear until Chapter 7, where they are introduced as a special case of the Z-transform (p. 61). It is stated that a Fourier series with coefficients in ℓ^2 converges in the mean, but this notion is not defined and the statement is not proved. Parseval's equation and the convolution theorem for Fourier series are also stated. The formula manipulation offered as justification is not adequate to prove the ℓ^2 version, but it does suffice for finite Fourier series, or trigonometric polynomials, which is all that the authors use. When they need to display frequency content, they plot either the absolute value of the Z-transform of a finite filter (a Laurent polynomial restricted to the unit circle) or the spectrogram of a finite sampled signal. The reader will have little difficulty reproducing the figures.

Where the text gets technical, it stays clear and honest. Chapters 7 and 12, entitled "Lifting and Filters I and II," are an expanded version of part of the famous paper by Daubechies and Sweldens [**2**]. The main theorem there is proved using Euclid's algorithm in the ring of Laurent polynomials. In my opinion, this is an excellent choice for an elementary exposition: the result and its proof are comprehensible and constructive, yet deep and nonobvious. It illuminates quite well how mathematics solves real problems.

There are sixty-five exercises and many figures, examples, and computer code fragments in this book. The computer codes are mostly base MATLAB scripts using function calls to a public software toolbox called *Uvi_Wave*. One filter transform (Function 5.2, p. 174) that is central to almost all the other algorithms is implemented in the C programming language, for speed. URLs for machine-readable versions of all codes are provided in Chapter 14. It and Chapter 13 are like appendices, with no exercises of their own.

I found some of the early exercises vague, and I imagine that student answers for these would be difficult to grade. For example, Exercises 4.1 and 6.1 are both, in essence, "Go through the examples in this chapter using MATLAB and *Uvi_Wave*." This reminded me of a text I used some years ago, with several exercises in each chapter beginning with "Write a report about...." Also, Chapter 3, which introduces the discrete wavelet transform, has only a single exercise. Instructors would need to pick and choose and perhaps create some new exercises for a real course.

There is evidence of haste in the copy-editing of this book, too, with the usual kind of typographical errors that result from word processing. Moreover, in the short bibliography, the first author of Oppenheim and Schafer's famous book *Digital Signal Processing* is listed as A. V. Oppenheimer. But this and the other bugs are unlikely to bother students.

3. THE BIG PICTURE. There are now many, many books on wavelets. In level of expected mathematical preparation, they range from informal introductions aimed at the general public such as Burke Hubbard [**5**] to detailed monographs for researchers such as the two-volume treatise of Meyer and Coifman [**7**], [**8**]. In devotion to applications, they range from aloof theoretical treatments such as Hernández and Weiss [**4**] to obsessive programming guides such as my own [**11**]. There are wavelet books tailored to various scientific and engineering specialties, too. Besides pure and applied mathematics books, there are compilations of introductory articles for many fields, such as Motard and Joseph [**9**] for chemical engineering, Foufoula–Georgiou and Kumar [**3**] for geophysics, and Akay [**1**] for biomedical signal engineering. Even some software systems such as *Mathematica*'s Wavelet Explorer, MATLAB's Wavelet Toolbox, and S-Plus Wavelets, have quite good introductions to wavelet signal processing in their documentation.

Were I faced with the choice of teaching a one-semester undergraduate course on wavelets with one of the books under review, which one would I choose? *Ripples*, I think, would be better for both me and the students. I might have to cook up a few extra exercises, but I could be quite sure that they would understand the mathematics and would learn to implement efficient wavelet transforms for time-frequency analysis with a computer. If the syllabus mandated that Fourier analysis be covered as well, then I would probably be forced to shortchange the students on proofs just like *A First Course* does, and with a little caution I could use it effectively, too.

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From Holomorphic Functions to Complex Manifolds. By Karl Fritzsche and Hans Grauert. Springer, New York, 2002, xv + 392 pp., ISBN 0-387-95395-7, $64.95.

Reviewed by **Steven G. Krantz**

When I was a graduate student in the early 1970s, it was difficult to learn the subject of several complex variables. There were few books. To be sure, the classics of Osgood [**14**], Bochner and Martin [**3**], Cartan [**4**], Fuks [**6**], Bers [**2**], Schwartz [**17**],