SIZE PROPERTIES OF WAVELET-PACKETS.

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1. Introduction

Wavelets are the building blocks of wavelet analysis in the same way as the functions $\cos nx$ are the building blocks of the ordinary Fourier analysis. But in contrast with sines and cosines, wavelets have a finite duration which can be arbitrarily small. This is the reason why the challenge of the construction of wavelets is to keep the best frequency localization which is allowed by Heisenberg's uncertainty principle.

The wavelet orthonormal basis with the best frequency localization was constructed in [**4**]. It is defined as the collection

(1.1) $$\psi_{j,k}(x) \;=\; 2^{j/2}\,\psi(2^j x - k) \quad , \quad j \in \mathbf{Z}\,,\; k \in \mathbf{Z}\,,$$

where ψ has the following properties

(1.2) $$\psi(x) \quad \text{belongs to the Schwartz class} \quad \mathcal{S}(\mathbf{R})$$

(1.3) $$\text{the Fourier transform} \quad \hat{\psi}(\xi) \quad \text{is supported by}$$
$$\frac{2\pi}{3} \;\le\; |\xi| \;\le\; \frac{8\pi}{3}$$

(1.4) $$\hat{\psi}(\xi) \;=\; e^{-\xi/2}\,\theta(\xi)$$

where $0 \le \theta(\xi) \le 1$ and $\theta(-\xi) = \theta(\xi)$

(1.5) $$\theta^2(2\pi - \xi) + \theta^2(\xi) \;=\; 1 \qquad \text{if} \quad \frac{2\pi}{3} \le \xi \le \frac{4\pi}{3}$$
$$\theta(2\xi) \;=\; \theta(2\pi - \xi) \qquad \text{if} \quad \frac{2\pi}{3} \le \xi \le \frac{4\pi}{3}\,.$$

The frequency localization of ψ is given by (1.3) while (1.4),(1.5) and (1.6) are convenient to provide an orthonormal sequence. The fact that this collection $\psi_{j,k}$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}$, is complete in $L^2(\mathbf{R}; dx)$ is, as often, related to some operator theory which will be described in section 2.

The Fourier transform $\hat{\psi}_{j,k}$ is supported by the "dyadic annulus" $\frac{2\pi}{3}2^j \leq |\xi| \leq \frac{8\pi}{3}2^j$ and this frequency localization is poor when j is large. Even if it means minor modification in the construction of ψ, one can achieve a slightly better frequency localization and replace $2\pi/3$ by $\pi - \delta$, $8\pi/3$ by $2\pi + 2\delta$. Then ψ_δ still belongs to the Schwartz class when $\delta > 0$ but the limiting case $\delta = 0$ gives the "Shannon wavelets" $\psi_0(x) = \frac{\sin 2\pi x}{2\pi x} - \frac{\sin \pi x}{\pi x}$. The relation with cardinal sines will be explained in section 4.

In some applications as speech signal processing one would like to be able to switch from a wavelet expansion to some orthonormal expansions offering a better frequency localization. This flexibility should not be ruined by the computational cost. In other words, most computations leading to wavelet coefficients should also provide the new coefficients. *Basic wavelet-packets* will be defined in section 4 and *general wavelet-packets* in section 8. They provide these new and efficient expansions (theorem 6). This remarkable efficiency is verified in numerical experiments on speech signal processing.

We would like to understand why wavelet-packets work so well and first to investigate their frequency localization. It will be proved (see theorem 3) that wavelet-packets do not enjoy the sharp frequency localization which has been announced in [**2**]. By S. Bernstein's inequalities, a sharp frequency localization would imply a uniform bound on L^∞-norms of the basic wavelet-packets $w_n(x)$. But theorem 3 shows that the average growth of $\|w_n\|_\infty$ is n^γ for some positive γ.

The fact that γ is rather small plays a key role in the construction of a large library of wavelet-packets orthonormal bases. Even if the problem of describing the full collection of such bases is still unsolved, the already known bases offer enough flexibility for the applications to speech signal processing.

2. The scaling function φ

In order to prove that the collection $\psi_{j,k}$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}$, is complete in $L^2(\mathbf{R})$, one tries to construct an approximation to the identity which is related to our wavelets.

This approximation to the identity will follow naturally from the following scheme.

DEFINITION 1. *A multiresolution analysis of $L^2(\mathbf{R})$ is an increasing sequence V_j, $j \in \mathbf{Z}$, of closed subspaces of $L^2(\mathbf{R})$ with the following properties*

$$\bigcap_{-\infty}^{\infty} V_j = \{0\} \quad , \quad \bigcup_{-\infty}^{\infty} V_j \quad \textit{is dense in } L^2(\mathbf{R}) \tag{2.1}$$

$$\forall\, f \in L^2(\mathbf{R}) \,,\ \forall\, j \in \mathbf{Z} \,, \qquad f(x) \in V_j \iff f(2x) \in V_{j+1} \tag{2.2}$$

$$\begin{aligned} &\textit{there exists a function} \quad \varphi \in \mathcal{S}(\mathbf{R}) \quad \textit{such that} \\ &\varphi(x-k) \,,\ k \in \mathbf{Z} \,, \quad \textit{is an orthonormal basis of } V_0. \end{aligned} \tag{2.3}$$

If we are given a multiresolution analysis, (2.2) implies

$$\frac{1}{2}\varphi\Big(\frac{x}{2}\Big) \;=\; \sum_{-\infty}^{\infty} \gamma_k \varphi(x+k) \tag{2.4}$$

where

$$\gamma_k \;=\; \int_{-\infty}^{\infty} \varphi\Big(\frac{x}{2}\Big)\, \overline{\varphi(x+k)}\, dx \;=\; 0(|k|^{-m})$$

for any $m \geq 1$.

Passing to the Fourier transform, one obtains

$$\hat{\varphi}(2\xi) \;=\; m_0(\xi)\hat{\varphi}(\xi) \quad , \quad m_0(\xi) \;=\; \sum_{-\infty}^{\infty} \gamma_k e^{ik\xi} \; . \tag{2.5}$$

We then define

$$m_1(\xi) \;=\; e^{-i\xi}\, \overline{m_0(\xi+\pi)} \tag{2.6}$$

and $\psi \in \mathcal{S}(\mathbf{R})$ by

$$\hat{\psi}(2\xi) \;=\; m_1(\xi)\, \hat{\varphi}(\xi) \; . \tag{2.7}$$

Denoting by W_j the orthogonal complement of V_j in V_{j+1} it is easy to check that

$$\psi(x-k)\;,\; k \in \mathbf{Z}\;, \quad \text{is an orthonormal basis of } W_0. \tag{2.8}$$

An obvious rescaling shows that $2^{j/2}\psi(2^j x - k)$, $k \in \mathbf{Z}$, is an orthonormal basis of W_j. Since $\bigcup_{-\infty}^{\infty} V_j$ is dense in $L^2(\mathbf{R})$, the full collection $\psi_{j,k}$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}$, is an orthonormal basis of $L^2(\mathbf{R})$.

It remains to be shown that the explicit ψ which is defined by (1.4) can also be obtained by (2.7). To prove this assertion, we define $\varphi \in \mathcal{S}(\mathbf{R})$ by the following conditions : $\varphi(-x) = \varphi(x)$, the Fourier transform $\hat{\varphi}(\xi)$ of $\varphi(x)$ is non-negative, $\hat{\varphi}(\xi) = 1$ on $[-2\pi/3\,,\, 2\pi/3]$ and

$$\sum_{-\infty}^{\infty} |\hat{\varphi}(\xi + 2k\pi)|^2 \;=\; 1 \; . \tag{2.9}$$

Condition (2.9) alone implies that $\varphi(x-k)$, $k \in \mathbf{Z}$, is an orthonormal sequence. This sequence spans a closed subspace denoted V_0. The other V_j's are defined by (2.2). It is easy to verify that all the other conditions in definition 1 are satisfied and that this algorithm leads to the function ψ as defined by (1.4).

3. Quadrature mirror filters

S. Mallat working on image processing made a fundamental discovery. He pointed out that some discrete algorithms named quadrature mirror filters (QMF's) were intimately related to multiresolution analysis (the latter concept was created by S. Mallat and one of the authors).

Quadrature mirror filters belong to a larger group of algorithms called *subband coding* which are used in speech processing as well as in image processing. The reader is referred to [**3**] or [**4**].

In our approach, a pair of quadrature mirror filters provides a dichotomy for every infinitely dimensional separable Hilbert space H, equipped with an orthonormal basis e_k, $k \in \mathbf{Z}$. A trivial dichotomy would be given by $H = H_0 \oplus H_1$ where H_0 is generated by (e_{2k}) and H_1 by (e_{2k+1}), $k \in \mathbf{Z}$. In a second example, H_0 is the closed linear span of the orthonormal sequence $\frac{e_{2k}+e_{2k+1}}{\sqrt{2}}$, $k \in \mathbf{Z}$, while H_1 is similarly spanned by $\frac{e_{2k}-e_{2k+1}}{\sqrt{2}}$, $k \in \mathbf{Z}$.

We now pass to the general case. Let (u_k) and (v_k) be two sequences in $l^2(\mathbf{Z})$. We consider the sequence (f_k) of vectors of H defined by

$$(3.1) \qquad \begin{cases} f_{2k} = \sum_{-\infty}^{\infty} u_{2k-l}\, e_l \\ f_{2k+1} = \sum_{-\infty}^{\infty} v_{2k-l}\, e_l \ . \end{cases}$$

We would like to know whether (f_k) is still an orthonormal basis of H. If so, $H = H_0 \oplus H_1$ where the sum is direct and orthonormal, H_0 being spanned by (f_{2k}) and H_1 by (f_{2k+1}).

We consider the following symbols

$$(3.2) \qquad \begin{cases} m_0(\theta) = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} u_k\, e^{ik\theta} \\ m_1(\theta) = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} v_k\, e^{ik\theta} \ , \end{cases}$$

and we have

Proposition 1.. *The three following properties are equivalent*

$$(f_k)_{k\in\mathbf{Z}} \quad \text{is an orthonormal sequence in } H \tag{3.3}$$

$$(f_k)_{k\in\mathbf{Z}} \quad \text{is an orthonormal basis of } H \tag{3.4}$$

$$\begin{aligned}&\text{for every } \theta\in[0,2\pi),\ \text{the matrix}\\ &S(\theta) \;=\; \begin{pmatrix} m_0(\theta) & m_1(\theta)\\ m_0(\theta+\pi) & m_1(\theta+\pi)\end{pmatrix} \quad \text{is unitary.}\end{aligned} \tag{3.5}$$

The first example corresponds to $m_0(\theta) = \frac{1}{\sqrt{2}}$ and $m_1(\theta) = \frac{1}{\sqrt{2}}\, e^{-i\theta}$. The second example to $m_0(\theta) = \frac{1}{2}(1+e^{i\theta})$, $m_1(\theta) = \frac{1}{2}(1-e^{i\theta})$.

We now consider the mapping $F = (F_0, F_1)$ which transforms the "old coordinates" (α_k) into the "new coordinates" β_{2k} and γ_{2k} as defined by the relation

$$\sum_{-\infty}^{\infty} \alpha_k e_k \;=\; \sum_{-\infty}^{\infty} \beta_{2k} f_{2k} + \sum_{-\infty}^{\infty} \gamma_{2k} f_{2k+1}\ . \tag{3.6}$$

We have $(\beta_{2k}) = F_0[(\alpha_k)]$ and $(\gamma_{2k}) = F_1[(\alpha_k)]$. The mapping F is a unitary isomorphism between $l^2(\mathbf{Z})$ and $l^2(2\mathbf{Z})\times l^2(2\mathbf{Z})$.

These two operators F_0 and F_1 will be called *quadrature mirror filters*.

4. Wavelets and quadrature mirror filters

Let us return to the multiresolution framework as defined in section 2. We have at our disposal two orthonormal bases for V_j, j being kept fixed. The first one is simply

$e_k = 2^{j/2}\varphi(2^j \cdot -k)$ while the second one is f_k where

$$(4.1) \qquad \begin{cases} f_{2k} &= 2^{(j-1)/2}\,\varphi(2^{j-1}\cdot -k) \\ f_{2k+1} &= 2^{(j-1)/2}\,\varphi(2^{j-1}\cdot -k)\,. \end{cases}$$

These two bases are connected by (3.1) if $m_0(\theta) = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} u_k e^{ik\theta}$ and $m_1(\theta) = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} v_k e^{ik\theta}$ are defined by

$$(4.2) \qquad \hat{\varphi}(2\xi) = m_0(\xi)\,\hat{\varphi}(\xi)$$

and

$$(4.3) \qquad \hat{\psi}(2\xi) = m_1(\xi)\,\hat{\varphi}(\xi)\,.$$

In other words, $m_0(\xi)$ is 2π -periodic, even, C^∞, non-negative, $m_0(\xi) = 1$ on $[-\pi/3, \pi/3]$ and in the end

$$(4.4) \qquad m_0^2(\xi) + m_0^2(\xi+\pi) = 1\,.$$

If $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ is replaced by $\left[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right]$ $\delta > 0$, the other properties of m_0 can be kept and this new $m_0(\theta)$ is closer to the ideal filter.

If $m_0(\xi) = 1$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $m_0(\xi) = 0$ on $[-\pi, \pi/2)$ and $[\pi/2, \pi)$, then $\hat{\varphi}(\xi) = 1$ on $[-\pi, \pi)$ and $\hat{\varphi}(\xi) = 0$ outside which gives $\varphi(x) = \frac{\sin \pi x}{\pi x}$. In that case V_j is the subspace of $L^2(\mathbf{R})$ defined by the condition that the Fourier transform of $f \in V_J$ is supported by $[-2^j, 2^j)$ and, in the same way, W_j is defined by the condition that $\hat{f}$ is supported by $2^j \le |\xi| < 2^{j+1}$. The price to be paid for this sharp frequency localization is the corresponding lack of localization of $\varphi(x)$ and $\psi(x)$ with respect to the x variable.

It should be noticed that (4.2) and $\hat{\varphi}(0) = 1$ imply

$$(4.5) \qquad \hat{\varphi}(\xi) = m_0(\xi/2)\,m_0(\xi/4)\,m_0(\xi/8)\ldots$$

Similarly we have

$$\hat{\psi}(\xi) \;=\; m_1(\xi/2)\, m_0(\xi/4)\, m_0(\xi/8) \dots \tag{4.6}$$

That leads to define $w_\varepsilon \in L^2(\mathbf{R})$ by

$$\hat{w}_\varepsilon(\xi) \;=\; m_{\varepsilon_1}(\xi/2)\, m_{\varepsilon_2}(\xi/4) \dots m_{\varepsilon_j}(\xi/2^j) \dots \tag{4.7}$$

when $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$, $\varepsilon_j \in \{0,1\}$ and $\varepsilon_j = 0$ when j is large enough. These functions $w_\varepsilon(x)$ will be our *basic wavelet-packets* and our goal is to investigate their properties.

An other approach to basic wavelet-packets will be proposed in section 5 and the L^p-norms of these basic wavelet-packets will be estimated in section 6 and 7 when p is large.

5. Definition of wavelet-packets

We consider two sequences (u_k) and (v_k) satisfying one of the equivalent conditions in proposition 1. It will be assumed that there exists a multiresolution analysis (V_j) of $L^2(\mathbf{R})$ which is connected to this pair of quadrature mirror filters by (2.5), (2.7) and (3.2). But we do not need more specific informations on the construction of φ and ψ.

The *basic wavelet-packets* $w_n(x)$, $n = 0, 1, 2, \dots$ are defined by the following recursion

$$w_{2n}(x) \;=\; \sqrt{2} \sum_{-\infty}^{\infty} u_k w_n(2x+k) \tag{5.1}$$

coupled with

$$w_{2n+1}(x) \;=\; \sqrt{2} \sum_{-\infty}^{\infty} v_k w_n(2x+k)\;, \tag{5.2}$$

the function $w_0(x)$ belonging to $L^1(\mathbf{R})$ and being normalized by

$$\int_{-\infty}^{\infty} w_0(x)\,dx \;=\; 1\,. \tag{5.3}$$

Let us start with $w_0(x)$. By (5.1), we have

$$w_0(x) \;=\; \sqrt{2}\sum_{-\infty}^{\infty} u_k w_0(2x+k) \tag{5.4}$$

and therefore

$$\hat{w}_0(2\xi) \;=\; m_0(\xi)\,\hat{w}_0(\xi) \tag{5.5}$$

$$\hat{w}_0(0) \;=\; 1\,. \tag{5.6}$$

But the unique continuous function satisfying (5.5) and (5.6) is $\hat{\varphi}(\xi)$ and therefore $w_0(x) = \varphi(x)$.

We now turn to (5.2) with $m = 0$. We obtain $w_1(x) = \psi(x)$. We can proceed and (5.1) gives $w_2(x)$. Then (5.2) gives $w_3(x)$ and so on...

Let us modify the labelling of the basic wavelet-packets. They will be labelled by the denumerable set E of all sequences $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$ where $\varepsilon_j \in \{0,1\}$ and $\varepsilon_j = 0$ eventually. Let $E_j \subset E$ be defined by $0 = \varepsilon_{j+1} = \varepsilon_{j+2} = \dots$. Then $E_j \uparrow E$. Finally the new labelling is given by $w_n(x) = w_\varepsilon(x)$ when $n = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{j-1}\varepsilon_j$.

Then the Fourier transform $\hat{w}_\varepsilon(\xi)$ of $w_\varepsilon(x)$ is given by

$$\hat{w}_\varepsilon(\xi) = m_{\varepsilon_1}(\xi/2)\,m_{\varepsilon_2}(\xi/4)\dots m_{\varepsilon_j}(\xi/2^j)\,\hat{\varphi}(\xi/2^j). \tag{5.7}$$

Since $\hat{\varphi}(\xi) = m_0(\xi/2)\,m_0(\xi/4)\dots$ and since $\varepsilon \in E_j$, (5.7) can be rewritten $\hat{w}_\varepsilon(\xi) = m_{\varepsilon_1}(\xi/2)\,m_{\varepsilon_2}(\xi/4)\dots$ and the fact that ε belongs to E_j can be ignored.

Wavelet-packets provide new orthonormal bases as theorem 1 shows.

THEOREM 1. *For each j, $j = 0, 1, 2 \dots$ the collection $w_\varepsilon(x-k)$, $\varepsilon \in E_j$, $k \in \mathbf{Z}$, is an orthonormal basis of V_j.*

Roughly speaking, theorem 1 means that the space V_j has been decoupled into 2^j orthonogonal channels $W^{(\varepsilon)}$, $\varepsilon \in E_j$. Since the band width of W_j, as defined by (1.3), is of the order of magnitude of 2^j, it was natural to expect the bandwidth of each $w_\varepsilon(x)$, $\varepsilon \in E_j$, to be $0(1)$. One of the goals of this work is to disprove this conjecture.

For proving theorem 1, we return to the labelling $n = 0, 1, \dots$ We want to prove that the collection

$$w_n(x-k) \quad , \quad 0 \le n < 2^j \quad , \quad k \in \mathbf{Z} \tag{5.8}$$

is an orthonormal basis of V_j.

When $j = 0$, we have $n = 0$, $w_0(x) = \varphi(x)$ and we know that $\varphi(x-k)$, $k \in \mathbf{Z}$, is an orthonormal basis of V_0. Let us assume that $w_n(x-k)$, $0 \le n < 2^{j-1}$, $k \in \mathbf{Z}$, is an orthonormal basis of V_{j-1}. Then (2.2) implies that $\sqrt{2}\, w_n(2x-k)$, $0 \le n < 2^{j-1}$, $k \in \mathbf{Z}$, is an orthonormal basis of V_j. But (5.1) and (5.2) can be rewritten

$$w_{2n}(x-k) = \sqrt{2} \sum_{-\infty}^{\infty} u_{2k-l}\, w_n(2x-l) \tag{5.9}$$

and

$$w_{2n+1}(x-k) = \sqrt{2} \sum_{-\infty}^{\infty} v_{2k-l}\, w_n(2x-l)\,. \tag{5.10}$$

This transformation is orthogonal since it has the same form as the one defined in (3.1). Therefore $w_n(x-k)$, $0 \le n < 2^j$, $k \in \mathbf{Z}$, is an orthonormal basis of V_j and theorem 1 is proved by induction on j.

Corollary. *The collection* $w_n(x-k)$, $n=0,1,2,\dots$, $k\in\mathbf{Z}$, *is an orthonormal basis of* $L^2(\mathbf{R})$.

6. L^∞-norms of wavelet-packets

Our goal is to study the frequency localization of the basic wavelet-packets $w_n(x)$. A convenient way for estimating this frequency localization is to compute

$$\sigma_n \;=\; \inf_{\xi_0\in\mathbf{R}} \int_{-\infty}^{\infty} |\xi-\xi_0|^2\,|\hat{w}_n(\xi)|^2\,\frac{d\xi}{2\pi} \tag{6.1}$$

Since $\int_{-\infty}^{\infty}|\hat{w}_n(\xi)|^2\,d\xi = 2\pi$, we have

$$\int_{-\infty}^{\infty} |\hat{w}_n(\xi)|\,d\xi \;\le\; \pi\sqrt{2+\sigma_n}\;. \tag{6.2}$$

But $\hat{w}_\varepsilon(\xi) \;=\; m_{\varepsilon_1}(\xi/2)\,m_{\varepsilon_2}(\xi/4)\cdots \;=\; e^{-i\lambda(\varepsilon)\xi}\;\; m_0(\xi/2+\varepsilon_1\pi)\;\; m_0(\xi/4+\varepsilon_2\pi)\dots$ where $\lambda(\varepsilon)=\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{4}+\dots$.

Since $0\le m_0(\xi)\le 1$, we obtain

$$\|w_\varepsilon\|_\infty \;=\; w_\varepsilon(\lambda(\varepsilon)) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{w}_\varepsilon(\xi)|\,d\xi \tag{6.3}$$

and therefore

$$\|w_n\|_\infty \;=\; \frac{1}{2}\sqrt{2+\sigma_n}\;. \tag{6.4}$$

That means that the growth of $\|w_n\|_\infty$ as n tends to infinity gives a lower bound of the frequency localization. In a still unpublished work, E. Séré assumed that $m_0(\xi)$ is strictly increasing on $\left[-\frac{2\pi}{3},-\frac{\pi}{3}\right]$ and satisfies the following condition

$$\sup_{\left\{-\frac{2\pi}{3}\le\xi\le-\frac{\pi}{3}\right\}} \big(m_0(\xi)+(\frac{\pi}{3}+\xi)m_0'(\xi)\big) \;=\; r \;<\; 2\,. \tag{6.5}$$

One can construct examples of 2π-periodic C^∞ functions $m_0(\xi)$ satisfying these two conditions and the ones mentioned above : (4.4), $m_0(\xi) = 1$ on $[-\pi/3, \pi/3]$ and $0 \leq m_0(\xi) \leq 1$.

Defining $\text{var}(\varepsilon)$ as $\sum_1^\infty |\varepsilon_{j+1} - \varepsilon_j|$, E. Séré proved the existence of two constants $\beta > \alpha > 1$, depending on $m_0(\xi)$, such that, for every $\varepsilon \in E$,

$$C_1 \alpha^{\text{var}(\varepsilon)} \;\leq\; \|w_\varepsilon\|_\infty \;\leq\; c_2 \beta^{\text{var}(\varepsilon)} \tag{6.6}$$

where $c_2 > c_1 > 0$ are two other constants.

Dropping (6.5), we want to prove a more general estimate.

THEOREM 2. *Let us assume that* $m_0(\xi) = 1$ *on* $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, $m_0(-\xi) = m_0(\xi)$, $0 \leq m_0(\xi) \leq 1$, $m_0^2(\xi) + m_0^2(\xi + \pi) = 1$ *and lastly*

$$m_0(\xi) \quad \textit{is decreasing on} \quad [0, \pi]\ . \tag{6.7}$$

Then we have (for $n \geq 1$*)*

$$\|w_n\|_\infty \;\leq\; C n^{1/4}\ . \tag{6.8}$$

Moreover if $m_0(\xi) = 1$ *on* $\left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right]$, $0 < \delta < \pi/2$, *we obtain*

$$\|w_n\|_\infty \;\leq\; C n^{\gamma(\delta)} \tag{6.9}$$

where $\gamma(\delta)$ *tends to 0 as* δ *tends to 0.*

The two proofs are similar and we begin with (6.8).

We already know that, if $\varepsilon \in E_j$, we have

$$\begin{aligned}
\|w_\varepsilon\|_\infty &= \frac{1}{2\pi} \int_{-\infty}^{\infty} m_0\big(\frac{\xi}{2} + \varepsilon_1\pi\big) \dots m_0\big(\frac{\xi}{2^j} + \varepsilon_j\pi\big) \hat{\varphi}\big(\frac{\xi}{2^j}\big)\, d\xi \\
&= \frac{2^j}{2\pi} \int_{-\infty}^{\infty} m_0(\xi + \varepsilon_j\pi) \dots m_0(2^{j-1}\xi + \varepsilon_1\pi)\, \hat{\varphi}(\xi)\, d\xi \\
&\le \frac{\sqrt{2}}{2\pi}\, 2^j \int_{-\pi}^{\pi} m_0(\xi + \varepsilon_j\pi) \dots m_0(2^{j-1}\xi + \varepsilon_1\pi)\, d\xi \\
&= \frac{\sqrt{2}}{2\pi}\, 2^j J(\varepsilon)\ .
\end{aligned}$$

This estimate follows from the fact that $\hat{\varphi}$ is compactly supported and m_0 is 2π-periodic.

Since $\hat{\varphi}(\xi) \ge \frac{1}{\sqrt{2}}$ on $[-\pi, \pi]$, we obtain the following two-sided estimate for $\varepsilon \in E_j$

$$\frac{2^j}{2\pi\sqrt{2}}\, J(\varepsilon) \ \le\ \|w_\varepsilon\|_\infty \ \le\ \frac{\sqrt{2}}{2\pi}\, 2^j J(\varepsilon)\ . \tag{6.10}$$

For estimating $J(\varepsilon)$, we apply the following observations (lemma 1).

Lemma 1. *If both $P(\xi)$ and $Q(\xi)$ are 2π-periodic and continuous functions of the real variable ξ, then, for any integer $q \ge 1$,*

$$\begin{aligned}
I \ =\ \int_{-\pi}^{\pi} P(\xi)\, Q(2^q\xi)\, d\xi \ &= \\
2^{-q} \int_{-\pi}^{\pi} \big[P(2^{-q}\xi) + \dots &+ P(2^{-q}\xi + 2^{-q}(2^q - 1)2\pi)\big]\, Q(\xi)\, d\xi
\end{aligned}$$

and therefore

$$|I| \ \le\ M\|Q\|_1 \tag{6.12}$$

where $\|Q\|_1 = \int_{-\pi}^{\pi} |Q(\xi)|\, d\xi$ and

$$M \ =\ 2^{-q} \sup_{0 \le \xi \le 2\pi} \big(|P(\xi)| + \dots + |P(\xi + 2^{-q}(2^q - 1)2\pi)|\big)\ .$$

This observation will be applied to $P(\xi) = m_0(\xi + \varepsilon_j\pi)\, m_0(2\xi + \varepsilon_{j-1}\pi)$, $Q(\xi) = m_0(\xi + \varepsilon_{j-2}\pi) \dots m_0(2^{j-3}\xi + \varepsilon_1\pi)$ and $q = 2$. We check by brute force that $M = \frac{\sqrt{2}}{4}$. Therefore

$$\int_{-\pi}^{\pi} m_0(\xi + \varepsilon_j\pi)\, m_0(2\xi + \varepsilon_{j-1}\pi) \dots m_0(2^{j-1}\xi + \varepsilon_1\pi)\, d\xi$$
$$\leq \frac{\sqrt{2}}{4} \int_{-\pi}^{\pi} m_0(\xi + \varepsilon_{j-2}\pi) \dots m_0(2^{j-2}\xi + \varepsilon_1\pi)\, d\xi$$

and an obvious induction gives (6.8).

To prove (6.9) we first consider the limiting case where $m_0(\xi)$ is replaced by the characteristic function $\chi_0(\xi)$ of $[-\pi/2, \pi/2)$. Then $\chi_0(\xi + \varepsilon_0\pi) \dots \chi_0(2^{q-1} + \varepsilon_{q-1}\pi)$, once restricted to $[-\pi, \pi)$, is either the characteristic function of $[-\pi 2^{-q}, \pi 2^{-q})$ or the characteristic function of the union $U(\varepsilon)$ of two intervals of length $\pi 2^{-q}$. If $0 < \delta < \pi 2^{-q-1}$ then the product $m_0(\xi + \varepsilon_0\pi) \dots m_0(2^{q-1}\xi + \varepsilon_{q-1}\pi)$ defines a bump function which is supported by $U(\varepsilon) + [-2\delta, 2\delta]$. It follows that the mean values of our product on $\xi_0 + 2k\pi 2^{-q}$, $0 \leq k < 2^q$, do not exceed $5 \cdot 2^{-q}$.

Returning to our problem of estimating $J(\varepsilon)$, we write $mq \leq j < (m+1)q$ where q will be frozen and m tends to infinity. Then lemma 1 implies $J(\varepsilon) = J_m(\varepsilon) \leq 5 \cdot 2^{-q} J_{m-1}(\varepsilon)$ and an obvious iteration gives $J(\varepsilon) \leq C 5^m \cdot 2^{-j}$. Finally, $\|w_\varepsilon\|_\infty \leq C 5^m$.

An optimal choice of q is to pick the largest integer such that $\delta < \pi 2^{-q-1}$. Therefore $\gamma(\delta) = 0(\log 1/\delta)^{-1}$.

Theorem 2 is now completely proved. The estimate given by (6.8) is sharp since, in a way, the reverse inequality is true, as theorem 3 shows.

THEOREM 3. *The assumptions on $m_0(\xi)$ being the same as in theorem 2, there exists a* *constant $r > 1$ such that, for $j \geq 0$,*

$$2^{-j} \sum_{\varepsilon \in E_j} \|w_\varepsilon\|_\infty \;\geq\; \frac{1}{\sqrt{2}} r^j \; . \tag{6.13}$$

To prove (6.12), we return to (6.10) and are led to estimating

$$\begin{aligned} S_j \;&=\; \sum_{\varepsilon \in E_j} \int_{-\pi}^{\pi} m_0(\xi + \varepsilon_1 \pi) \ldots m_0(2^{j-1}\xi + \varepsilon_j \pi)\, d\xi \\ &=\; \int_{-\pi}^{\pi} \sigma(\xi)\sigma(2\xi) \ldots \sigma(2^{j-1}\xi)\, d\xi \end{aligned}$$

where

$$\sigma(\xi) \;=\; m_0(\xi) + m_0(\xi + \pi) \;\geq\; 1 \; .$$

But

$$\begin{aligned} &\log \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma(t)\sigma(2t) \ldots \sigma(2^{j-1}t)\, dt \right\} \\ &\geq\; \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \{\sigma(t) \ldots \sigma(2^{j-1}t)\}\; dt \\ &=\; \frac{j}{2\pi} \int_{-\pi}^{\pi} \log\, \sigma(t)\, dt \;=\; \beta j \qquad \text{where} \quad \beta > 0 \; . \end{aligned}$$

If $m_0(\xi) = 1$ on $\left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right]$, then $\sigma(t) \geq 1$ on $\left[\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta\right]$ and in general β will be of the order of magnitude of $C\delta$. Finally the average lower bound of $\|w_n\|_\infty$ is $n^{\beta(\delta)}$ where $\beta(\delta)$ tends to 0 as δ tends to 0.

7. L^p-norms of wavelet-packets

Theorem 4. *Let us keep the notations and assumptions of theorem 2 and theorem 3. Then there exists a $p_0 \geq 2$ such that, for each $p > p_0$, one can find a positive $\gamma = \gamma(p)$*

with the property that

$$\varlimsup_{n\to+\infty} n^{-\gamma}\big(\|w_1\|_p + \cdots + \|w_n\|_p\big) \;>\; 0\,. \tag{7.1}$$

In other words, the average growth of $\|w_n\|_p$ is $n^{\gamma(p)}$ where $\gamma(p) > 0$ when p is large. It means that $w_n(x)$ cannot be a product $u_n(x)v_n(x)$ between some highly oscillating bounded factor $u_n(x)$ and some envelope $v_n(x)$ which would keep a given shape with bounded sizes.

Theorem 4 easily follows from S. Bernstein's inequalities. When $\varepsilon \in E_j$, the Fourier transform $\hat{w}_\varepsilon(\xi)$ of $w_\varepsilon(x)$ is supported by the interval $|\xi| \le \frac{8\pi}{3}2^j$. Bernstein's inequality gives

$$\|w_\varepsilon\|_\infty \;\le\; C2^{j/p}\,\|w_\varepsilon\|_p \tag{7.2}$$

and since the average value of $\|w_\varepsilon\|_\infty$ is large, so is $\|w_\varepsilon\|_p$ as long as $2^{1/p} < r$ (and therefore p_0 tends to ∞ as δ tends to 0).

8. Other orthonormal bases

Let us begin with the description of a rather general splitting scheme indexed by a dyadic tree. Let us fix two sequences (u_k) and (v_k) defining a pair of quadrature mirror filters, as in section 3. We start with a Hilbert space H equipped with a given orthonormal basis $(e_k)_{k\in\mathbf{Z}}$ and we split H accordingly to (3.1). Let us write $e_k^{(0)} = f_{2k}$ and $e_k^{(1)} = f_{2k+1}$. We now consider H_0 equipped with $e_k^{(0)}$, $k \in \mathbf{Z}$, and we go on splitting H_0 with the same sequences u_k and v_k. We obtain two new subspaces $H_{0,0}$ and $H_{0,1}$ equipped with the corresponding orthonormal bases $e_k^{(0,0)}$ and $e_k^{(0,1)}$ as defined by (3.1). Similarly H_1 is split into $H_{1,0} \oplus H_{1,1}$.

At the j-th step we have obtained 2^j subspaces H_α, $\alpha \in \{0,1\}^j$, of H. A very convenient notation will be to write $H_\alpha = H_I$ where I is the dyadic interval $\left[\frac{\alpha_1}{2} + \cdots + \frac{\alpha_j}{2^j}\ ,\ \frac{\alpha_1}{2} + \cdots + \frac{\alpha_j}{2^j} + \frac{1}{2^j}\right)$ when $\alpha = (\alpha_1, \ldots, \alpha_j)$. This labelling has the following advantage. If I is a dyadic interval (contained in $[0,1)$) and if $I = I_1 \cup I_2 \cup \cdots \cup I_m$ is a partition of I by dyadic intervals, then

$$H_I \ = \ H_{I_1} \oplus \cdots \oplus H_{I_m} \tag{8.1}$$

the sum being direct and orthogonal.

Does this identity still hold when an infinite sequence I_m, $m = 1, 2, \ldots$ of dyadic intervals forms a partition of I ? If this is true we can raise the more difficult problem where I, except for a null set, is covered by a sequence I_m of disjoint dyadic intervals. By null set we mean either a null set with respect to the Lebesgue measure or to some other measure adapted to the given quadrature mirror filters.

A first answer is given by the following theorem.

Theorem 5. *Let us assume that, except for a denumerable set, a dyadic interval I is covered by the union $\bigcup_1^\infty I_m$ of disjoint dyadic intervals I_m, $m = 1, 2, \ldots$.*

Let us also assume that $m_0(\theta) = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} u_k e^{ik\theta}$ satisfies the same hypothesis as in theorem 2. Then

$$H_I \ = \ H_{I_1} \oplus H_{I_2} \oplus \cdots \oplus H_{I_m} \oplus \ldots$$

where the sum is direct and orthogonal.

Before proving theorem 5, let us give an application. The basic wavelet-packets are $w_n(x-k)$, $n = 0, 1, 2, \ldots$, $k \in \mathbf{Z}$, and the *general wavelet-packets* will be defined as

$$2^{q/2}\, w_n(2^q x - k) \quad , \quad n \in \mathbf{N}\ ,\ q \in \mathbf{Z}\ ,\ k \in \mathbf{Z}\ . \tag{8.2}$$

This full collection clearly is an over complete system in $L^2(\mathbf{R})$ and our goal will be to construct orthonormal bases of $L^2(\mathbf{R})$ with sub-collections of the form

$$2^{q/2}\, w_n(2^q x - k) \quad , \quad k \in \mathbf{Z}\ ,\ (n,q) \in E\ .$$

An obvious solution is given by $q = 0$, $n = 0, 1, \dots$ and an other one by $n = 1$ and $q \in \mathbf{Z}$.

To describe some other possibilities, let us associate the dyadic interval $I(n,q) = [2^q n, 2^q(n+1))$ to each of the wavelet-packets $2^{q/2} w_n(2^q x - k)$, $k \in \mathbf{Z}$.

We then have

THEOREM 6. *If a subset $E \subset \mathbf{N} \times \mathbf{Z}$ has the property that, except for a denumerable set, $[0,\infty)$ is covered by the disjoint union of the dyadic intervals $I(n,q)$, $(n,q) \in E$, then the corresponding wavelet-packets*

$$2^{q/2}\, w_n(2^q x - k) \quad , \quad k \in \mathbf{Z}\ ,\ (n,q) \in E\ , \tag{8.3}$$

form an orthonormal basis of $L^2(\mathbf{R})$.

Theorem 6 can be deduced from theorem 5 if the following simple remark which has already been used is kept in mind. We first identify the abstract Hilbert space H equipped with an orthonormal basis $(e_k)_{k\in\mathbf{Z}}$ to the space V_N equipped with the orthonormal basis $2^{N/2}\varphi(2^N x - k)$, $k \in \mathbf{Z}$. We denote by E_N the subset of E defined by $2^{-N} I(n,q) \subset I = [0,1)$. If the intervals I_m appearing in theorem 5 are precisely these $2^{-N} I(n,q)$, $(n,q) \in E_N$, then theorem 5 states that the collection

$$2^{q/2} w_n(2^q x - k) \quad , \quad k \in \mathbf{Z}\ ,\ (n,q) \in E_N \tag{8.4}$$

is an orthonormal basis of V_N.

It suffices to let N tend to infinity to obtain theorem 6.

THE PROOF OF THEOREM 5:

To prove theorem 5 as stated, it suffices to consider the case where $I = [0,1)$ and $H_I = H$. We denote by $\pi_m : H \to H_{I_m}$ the orthogonal projector and we want to show that, for each $x \in H$, we have

$$\|w\|^2 \;=\; \sum_1^\infty \|\pi_m(x)\|^2 \;. \tag{8.5}$$

This relation will ensure that H is the closed linear span of the orthogonal subspaces H_{I_m}.

To prove (8.5), we consider a given $x \in H$ and without losing generality we can assume $\|x\| = 1$. If I is any dyadic subinterval of $[0,1)$, we write $\omega(I) = \|\pi_I(x)\|^2$ where $\pi_I(x)$ is the orthogonal projection of x on H_I. This functional ω is finitely additive and we want to extend this property and prove

$$\omega(I) \;=\; \omega(I_1) + \omega(I_2) + \cdots + \omega(I_m) + \ldots \tag{8.6}$$

when, except for a denumerable set, I is the union $\bigcup_1^\infty I_m$ of the pairwise disjoint dyadic intervals I_m.

The following lemma 2 will immediately imply (8.5).

LEMMA 2. *For $x \in H$, $\|x\| = 1$, there exists a continuous measure μ on $[0,1)$ such that, for every dyadic interval $I \subset [0,1)$,*

$$\|\pi_I(x)\|^2 \;=\; \int_I d\mu(t) \;. \tag{8.7}$$

To prove the existence of μ, it suffices to check the following continuity property of the additive set functional $\omega(I)$:

$$\begin{array}{l} \textit{if} \quad I^{(1)} \supset I^{(2)} \supset \cdots \supset I^{(j)} \supset \ldots \quad \textit{where the length} \\ \textit{of} \quad I^{(j)} \quad \textit{is} \quad 2^{-j} \quad, \textit{then} \quad \omega(I^{(j)}) \quad \textit{tends to} \quad 0 \;. \end{array} \tag{8.8}$$

We will show a more precise estimate.

LEMMA 3. *For* $x = \sum_{-N}^{N} \xi_k e_k$, $N = 1, 2, \dots$ *and* $\|x\| = 1$, *we have*

$$\|\pi_I(x)\| \;\leq\; CN^2 |I|^{1/4} \tag{8.9}$$

where C *is an absolute constant and* $|I|$ *denotes the length of* I.

If we admit lemma 3, (8.7) follows easily. Assuming $\|x\| = 1$, we write $x = \sum_{-\infty}^{\infty} \xi_k e_k$ and denote by x_N the finite sum $\sum_{-N}^{N} \xi_k e_k$. We then have $\|\pi_I(x)\| = \|\pi_I(x - x_n)\| + \|\pi_I(x_n)\| \leq \|x - x_N\| + CN^2|I|^{1/4} \leq \varepsilon$

whenever $\|x - x_N\| \leq \varepsilon/2$ (which fixes N) and then $CN^2|I|^{1/4} \leq \varepsilon/2$ (which gives $|I| \leq \eta(\varepsilon)$).

To prove lemma 3, we will use a specific realization of H and of the corresponding subspaces H_I. There are several (isometrically equivalent) such realizations and our choice will be dictated by convenience.

We consider the realization where H is $L^2\big([0, 2\pi], \frac{d\theta}{2\pi}\big)$ and where e_k becomes $e^{-ik\theta}$, $k \in \mathbf{Z}$. Then the vectors f_{2k} will be $\sqrt{2}\, m_0(\theta)\, e^{-2ik\theta}$ and f_{2k+1} will be $\sqrt{2}\, m_1(\theta)\, e^{-2ik\theta}$. We can proceed further and finally the orthonormal basis of H_I will be

$$2^{j/2} m_{\varepsilon_0}(\theta) \dots m_{\varepsilon_{j-1}}(2^{j-1}\theta)\, e^{-i2^j k\theta}$$

when

$$I \;=\; \left[\frac{\varepsilon_0}{2} + \cdots + \frac{\varepsilon_{j-1}}{2^j}\,,\; \frac{\varepsilon_0}{2} + \cdots + \frac{\varepsilon_{j-1}}{2^j} + \frac{1}{2^j}\right).$$

Finally our vector $x = \sum_{-N}^{N} \xi_k e_k$ will be a trigonometric polynomial f and

$$
\begin{aligned}
\|\pi_I(f)\|^2 &= 2^j \sum_k \left| \int_0^{2\pi} f(\theta) m_{\varepsilon_0}(\theta) \dots m_{\varepsilon_{j-1}}(2^{j-1}\theta)\, e^{i2^j k\theta} \frac{d\theta}{2\pi} \right|^2 \\
&= 2^j \sum_k |\lambda(j,k)|^2
\end{aligned}
\tag{8.10}
$$

We need exactly the same estimate as the one in theorem 2 :

$$
\int_0^{2\pi} |m_{\varepsilon_1}(\theta) \dots m_{\varepsilon_j}(2^{j-1}\theta)|\, d\theta \;\leq\; C 2^{-j} 2^{j/4} \,. \tag{8.11}
$$

As we know this estimate is not optimal and the factor $2^{j/4}$ can be replaced by $2^{\gamma(\delta)j}$ depending on the properties of $m_0(\theta)$.

If $k = 0$, $|\lambda(j,0)| \leq C\|f\|_\infty\, 2^{-j}\, 2^{j/4}$ which is the required bound.

When $k \neq 0$, we integrate by parts and rewrite

$$
\lambda(j,k) = - i(k2^j)^{-1} \int_0^{2\pi} \frac{d}{d\theta} \left\{ f(\theta) m_{\varepsilon_0}(\theta) \dots m_{\varepsilon_{j-1}}(2^{j-1}\theta) \right\} e^{i2^j k\theta} \frac{d\theta}{2\pi} \,.
$$

The term where $f(\theta)$ is differentiated is treated as above. The term

where $m_{\varepsilon_q}(2^q\theta)$ is differentiated $(0 \leq q < j)$ is bounded by $C|k|^{-1}\, 2^{q-j}\, \lambda(q,j,k)$ where

$$
\lambda(q,j,k) = \int_0^{2\pi} \left| m_{\varepsilon_0}(\theta) \dots m_{\varepsilon_{q-1}}(2^{q-1}\theta) \right| \left| m_{\varepsilon_{q+1}}(2^{q+1}\theta) \dots m_{\varepsilon_{j-1}}(2^{j-1}\theta) \right| d\theta \,.
$$

To estimate this integral, we return to the argument used in theorem 2. We group the first q terms (or $q-1$ terms if q is odd) by pairs and apply lemma 1 inductively. We gain a factor $(\sqrt{2}/4)^{(q-1)/2}$. We then repeat this treatment on the second half of the integrand, starting from $m_{\varepsilon_{q+1}}(2^{q+1}\theta)$. We obtain a factor $(\sqrt{2}/4)^{(j-q-1)/2}$. All together, we have obtained $C(\sqrt{2}/4)^{j/2}$ and the sum over q gives $\frac{C}{|k|}(\sqrt{2}/4)^{j/2}$. Finally the l^2 norm of this sequence is $C'(\sqrt{2}/4)^{j/2}$ as announced.

Theorem 5 is completely proved.

It would be interesting to know whether all the measures $\mu = \mu_x$, $x \in H$, are absolutely continuous with respect to the Lebesgue measure on $[0,1)$. In that case, theorem 5 could be extended to the situation where except for a null set, $[0,1)$ is covered by $\bigcup_1^\infty I_m$.

The study of these measures μ_x can be simplified by the following remark.

Lemma 4.

$$\|\mu_x - \mu_y\| \ \leq \ \|x-y\|(\|x\| + \|y\|) \ .$$

For proving this estimate, it suffices to write

$$\|\mu_x - \mu_y\| \ = \ \lim_{m \uparrow +\infty} \sum_{|I|=2^{-m}} |\mu_x(I) - \mu_x(I)|$$

where the sum runs over all dyadic intervals $I \subset [0,1)$ with length 2^{-m}. But

$$\begin{aligned} |\mu_x(I) - \mu_y(I)| \ &= \ \big| \|\pi_I(x)\|^2 - \|\pi_I(y)\|^2 \big| \\ &= \ \big| \|\pi_I(x)\| - \|\pi_I(y)\| \big| \ \big(\|\pi_I(x)\| - \|\pi_I(y)\| \big) \\ &\leq \ \|\pi_I(x)\| \, \|\pi_I(x-y)\| + \|\pi_I(y)\| \, \|\pi_I(x-y)\| \ . \end{aligned}$$

The two terms are similar and the Cauchy-Schwarz inequality applied to

$$\sum_{|I|=2^{-m}} \|\pi_I(x)\| \, \|\pi_I(x-y)\|$$

gives

$$\Big(\sum_{|I|=2^{-m}} \|\pi_I(x)\|^2 \Big)^{1/2} \Big(\sum_{|I|=2^{-m}} \|\pi_I(x-y)\|^2 \Big)^{1/2} \ = \ \|x\| \, \|x-y\| \ .$$

9. Conclusion

To conclude, we describe an example showing that theorem 5 or theorem 6 do not give the final answer to the problem which has been raised. More precisely we show that some Cantor sets may play the role of the exceptional denumerable set. We denote by $K \subset [0,1]$ a symmetric Cantor set with dissection ratio $1/4$. It means that K can be covered by 2^j intervals of length 4^{-j} and it will be the only property we shall use. Consider the open intervals $]a_m, b_m[$ which are the components of $[0,1] \setminus K$ and assume that $I_m = [a_m, b_m)$ in theorem 5 or 6. Returning to theorem 2, let us assume that $\|w_n\|_\infty \leq Cn^\gamma$ with $0 < \gamma < 1/4$. We know that this can be achieved. Then the reasoning which was used for theorem 5 gives $\mu(I) \leq C2^{-(1-2\gamma)j}$ for each dyadic interval I contained in $[0,1)$ with length 2^{-j}. Finally $\mu(K) = 0$ and the conclusions of theorem 5 or 6 are valid.

This example shows that there exist some wavelet-packets orthonormal bases far beyond the ones described in theorem 6. It also shows that this fact is related to the slow growth of $\|w_n\|_\infty$ is announced in the introduction.

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