# SMOOTH LOCALIZED ORTHONORMAL BASES

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Abstract. We describe a decomposition of $L^2(\mathbf{R})$ into an orthogonal direct sum of copies of $L^2(\mathbf{T})$. The decomposition maps smooth functions to smooth periodic functions. It generalizes certain earlier constructions of smooth orthonormal windowed bases. In particular, it shows the existence of smooth orthonormal windowed exponential, wavelet, and wavelet packet bases for $L^2(\mathbf{R})$.

## 1. Introduction

Orthogonal projections which map smooth functions to smooth compactly supported functions appeared in the work of Malvar [M] and Coifman and Meyer [CM]. In those papers the projections were used to build a smooth overlapping orthogonal basis on the line, composed of windowed sine (or cosine) functions. In this paper we observe that a variation of the method provides a smooth orthogonal projection onto periodic functions. Conversely, it permits arbitrary smooth periodic bases to be used as smooth "windowed" bases on the line. It evades the Balian–Low obstruction by a modification of the definition of "window."

Many of these bases' properties were only briefly described in the short papers of Malvar, Coifman, and Meyer, but are developed in detail in [AWW]. We do not wish to overlook the original sources, but we will take advantage of some of the later paper's structure and notation for purely pedagogical reasons.

## 2. Smooth Orthogonal Projections

The main ingredient in the recipe is a pair of orthogonal projections which can be factored into simple pieces. Let $r = r(t)$ be a function in the class $C^d(\mathbf{R})$ for some $0 \le d \le \infty$, satisfying the following conditions:

$$|r(t)|^2 + |r(-t)|^2 = 1 \quad \text{for all } t \in \mathbf{R}; \qquad r(t) = \begin{cases} 0, & \text{if } t \le -1, \\ 1, & \text{if } t \ge 1; \end{cases} \tag{1}$$

An example function $r \in C^1$ is the following:

$$r(t) = \begin{cases} 0, & \text{if } t \le -1, \\ \sin\left[\frac{\pi}{4}(1 + \sin\frac{\pi t}{2})\right], & \text{if } -1 < t < 1, \\ 1, & \text{if } t \ge 1. \end{cases} \tag{2}$$

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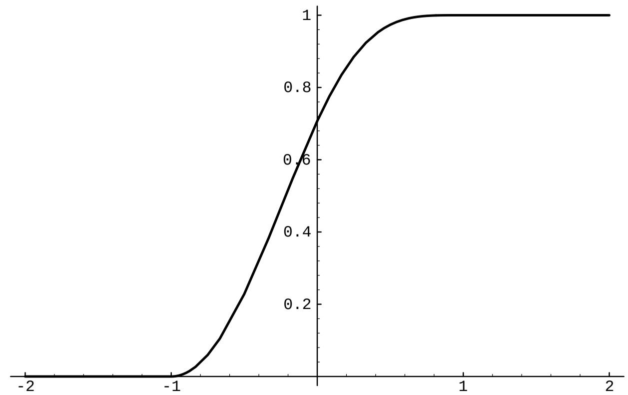


**Figure 1.**

**Example of $C^1$ cutoff function.**

A general construction for such functions is given in [AWW]. Now define the *folding* operator $U = U(r)$ and its adjoint *unfolding* operator $U^* = U^*(r)$:

$$(3) \qquad Uf(t) = \begin{cases} r(t)f(t) + r(-t)f(-t), & \text{if } t > 0, \\ \overline{r(-t)}f(t) - \overline{r(t)}f(-t), & \text{if } t < 0; \end{cases}$$

$$(4) \qquad U^*f(t) = \begin{cases} \overline{r(t)}f(t) - r(-t)f(-t), & \text{if } t > 0, \\ r(-t)f(t) + \overline{r(t)}f(-t), & \text{if } t < 0. \end{cases}$$

Observe that $Uf(t) = f(t)$ and $U^*f(t) = f(t)$ if $t \geq 1$ or $t \leq -1$. Also, $U^*Uf(t) = UU^*f(t) = \left(|r(t)|^2 + |r(-t)|^2\right)f(t) = f(t)$ for all $t \neq 0$, so that $U$ and $U^*$ are unitary isomorphisms of $L^2(\mathbf{R})$. As an example we compute $Uf$ in a particularly simple case, using the cutoff function defined in Eq.(2):

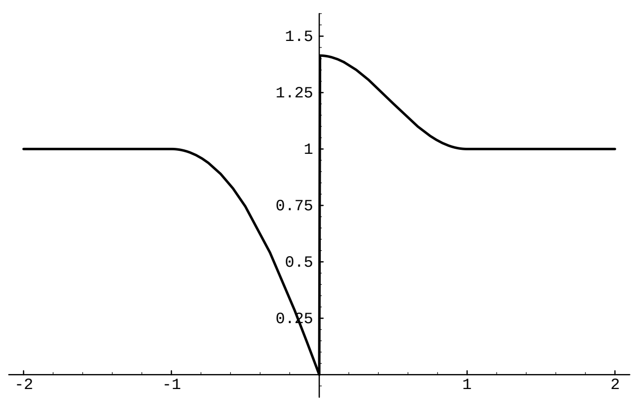


**Figure 2.**

**Action of $U$ on the constant function $f(t) = 1$.**

It does not matter how we define $Uf(0)$ or $U^*f(0)$ for functions $f \in L^2$; for smooth $f$ we may just as well define $Uf(0) \stackrel{\text{def}}{=} f(0)$, and for $f$ satisfying certain smoothness and boundary limit conditions we will show that there is a unique smooth extension of $U^*f$ across $t = 0$.

**Lemma 1.** *Suppose $r \in C^d(\mathbf{R})$ for $0 \le d \le \infty$. If $f \in C^d(\mathbf{R})$, then $Uf$ has $d$ continuous derivatives in $\mathbf{R} \setminus \{0\}$, and for all $0 \le n \le d$ there exist limits $[Uf]^{(n)}(0+)$ and $[Uf]^{(n)}(0-)$ which satisfy the following conditions:*

$$\begin{aligned} \lim_{t \to 0+} [Uf]^{(n)}(t) &= 0 \qquad \text{if } n \text{ is odd,} \\ \lim_{t \to 0-} [Uf]^{(n)}(t) &= 0 \qquad \text{if } n \text{ is even.} \end{aligned} \tag{5}$$

*Conversely, if $f$ belongs to $C^d(\mathbf{R} \setminus \{0\})$ and has limits $f^{(n)}(0+)$ and $f^{(n)}(0-)$ for all $0 \le n \le d$ which satisfy the equations*

$$\begin{aligned} \lim_{t \to 0+} f^{(n)}(t) &= 0 \qquad \text{if } n \text{ is odd,} \\ \lim_{t \to 0-} f^{(n)}(t) &= 0 \qquad \text{if } n \text{ is even,} \end{aligned} \tag{6}$$

*then $U^* f$ has a unique continuous extension (across $t = 0$) which belongs to $C^d(\mathbf{R})$.*

*Proof.* The smoothness of $Uf$ and $U^* f$ on $(0, \infty)$ and $(-\infty, 0)$ follows from elementary calculus. We can calculate the one-sided limits of the derivatives as follows:

$$\begin{aligned} \lim_{t \to 0+} [Uf]^{(n)}(t) &= \lim_{t \to 0+} \sum_{k=0}^{n} \binom{n}{k} \left[ r^{(n-k)}(t) f^{(k)}(t) + (-1)^n r^{(n-k)}(-t) f^{(k)}(-t) \right] \\ &= \sum_{k=0}^{n} \binom{n}{k} \left[ r^{(n-k)}(0) f^{(k)}(0+) + (-1)^n r^{(n-k)}(0) f^{(k)}(0-) \right] \end{aligned} \tag{+}$$

$$\begin{aligned} \lim_{t \to 0-} [Uf]^{(n)}(t) &= \lim_{t \to 0-} \sum_{k=0}^{n} \binom{n}{k} \left[ (-1)^{n-k} \overline{r^{(n-k)}(-t)} f^{(k)}(t) - (-1)^k \overline{r^{(n-k)}(t)} f^{(k)}(-t) \right] \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[ (-1)^n \overline{r^{(n-k)}(0)} f^{(k)}(0-) - \overline{r^{(n-k)}(0)} f^{(k)}(0+) \right] \end{aligned} \tag{$-$}$$

If $n$ is odd, with $0 \le n \le d$, then the summands in the right-hand side of Eq.(+) are $r^{(n-k)}(0)[f^{(k)}(0+) - f^{(k)}(0-)] = 0$, since $f^{(k)}$ is continuous at 0 for all $0 \le k \le d$. If $n$ is even, then the summands in the right-hand side of Eq.($-$) are $r^{(n-k)}(0)[f^{(k)}(0-) - f^{(k)}(0+)] = 0$ for the same reason.

The converse requires showing the equality of two one-sided limits:

$$\begin{aligned} \lim_{t \to 0+} [U^* f]^{(n)}(t) - \lim_{t \to 0-} [U^* f]^{(n)}(t) &= [U^* f]^{(n)}(0+) - [U^* f]^{(n)}(0-) \\ &= \sum_{k=0}^{n} \binom{n}{k} \left[ \overline{r^{(n-k)}(0)} f^{(k)}(0+) - (-1)^n r^{(n-k)}(0) f^{(k)}(0-) \right. \\ &\qquad \left. - (-1)^{n-k} r^{(k)}(0) f^{(k)}(0-) - (-1)^k \overline{r^{(n-k)}(0)} f^{(k)}(0+) \right] \\ &= \sum_{k=0}^{n} \binom{n}{k} \left[ \left\{ 1 - (-1)^k \right\} \overline{r^{(n-k)}(0)} f^{(k)}(0+) - (-1)^n \left\{ 1 + (-1)^k \right\} r^{(n-k)}(0) f^{(k)}(0-) \right] \end{aligned} \tag{7}$$

The right-hand side is zero, since both $\left\{ 1 - (-1)^k \right\} f^{(k)}(0+)$ and $\left\{ 1 + (-1)^k \right\} f^{(k)}(0-)$ vanish for all $k$. Since the one-sided limits agree, we know that $\lim_{t \to 0} [U^* f]^{(n)}(t)$ exists for $0 \le n \le d$. Now the function

$U^*f$ has a unique continuous extension across $t = 0$. By the mean value theorem, for each $t \neq 0$ there is some $t_0$ between 0 and $t$ such that

$$\frac{[U^*f]^{(k)}(t) - [U^*f]^{(k)}(0)}{t} = [U^*f]^{(k+1)}(t_0) \tag{8}$$

By letting $t \to 0$ in this equation, we show that $[U^*f]^{(k)}(0) = \lim_{t\to 0}[U^*f]^{(k)}(t) \implies [U^*f]^{(k+1)}(0) = \lim_{t\to 0}[U^*f]^{(k+1)}(t)$ for $0 \leq k < d$. Induction on $k$ then shows that the unique continous extension of $U^*f$ belongs to $C^d(\mathbf{R})$. □

This lemma shows that just a trivial boundary condition is needed to obtain smoothness. In particular, the 0 function satisfies the condition, and we shall use this fact to clarify the construction of the smooth orthogonal projections in [AWW]. Recall that these were defined directly in terms of the cutoff function:

$$P_0 f(t) = |r(t)|^2 f(t) + \overline{r(t)}r(-t)f(-t); \qquad P^0 f(t) = |r(-t)|^2 f(t) - \overline{r(t)}r(-t)f(-t). \tag{9}$$

We can relate $P_0, P^0$ to some trivial orthogonal projections given by restriction to intervals, as defined below:

$$\chi_I f(t) = \begin{cases} f(t), & \text{if } t \in I, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Then $P^0$ and $P_0$ may be defined as operators on $L^2(\mathbf{R})$ by the following formulas:

$$P_0 = U^*\chi_{\mathbf{R}^+}U; \qquad P^0 = U^*\chi_{\mathbf{R}^-}U \tag{11}$$

Since the operators $P_0, P^0$ are obtained from the trivial orthogonal projections by unitary conjugation, they are themselves orthogonal projections. We call them *smooth projections* onto half-lines because of the following properties:

**Corollary 2.** *If $f \in C^d(\mathbf{R})$, then the unique continuous extensions of $P_0 f$ and $P^0 f$ belong to $C^d(\mathbf{R})$, and* $\operatorname{supp} P^0 f \subset (-\infty, 1]$ *and* $\operatorname{supp} P_0 f \subset [-1, \infty)$.

*Proof.* The result follows from a two-way application of Lemma 1, since $\chi_{\mathbf{R}^+}Uf(t)$ and $\chi_{\mathbf{R}^-}Uf(t)$ satisfy Eq.(6) at $t = 0$. □

It is evident from either formula that $P_0 + P^0 = I$ and that both $P_0$ and $P^0$ are selfadjoint. We remark that these formulas follow the "local cosine" polarity, and that we can just as well exchange the $+$ and $-$ in the definitions of $P^0$ and $P_0$ (likewise for $U$ and $U^*$) to obtain the "local sine" polarity.

Now consider the usual translation and rescaling operators:

$$\begin{aligned} \tau_\alpha f(t) &= f(t+\alpha); & \tau_\alpha^* f(t) &= f(t-\alpha); \\ \delta_\epsilon f(t) &= \sqrt{\epsilon} f(\epsilon t); & \delta_\epsilon^* f(t) &= \frac{1}{\sqrt{\epsilon}} f(t/\epsilon). \end{aligned} \tag{12}$$

Here $\alpha$ and $\epsilon > 0$ are real numbers. The "range of influence" of the folding and projection operators can be dilated and translated to an arbitrary interval $(\alpha-\epsilon,\ \alpha+\epsilon)$ by conjugation with $\delta_\epsilon$ and $\tau_\alpha$. We can also use an arbitrary smoothly rising cutoff function $r$ as long as it satisfies the conditions in Eq.(1).

We can now define smooth orthogonal projections onto compactly-supported functions in the notation of [AWW]. We dilate $P^0$ and $P_0$ by conjugation with $\delta_\epsilon$ and then translate by conjugation with $\tau_\alpha$:

$$P_{\alpha\epsilon} = \tau_\alpha^* \delta_\epsilon^* P_0 \delta_\epsilon \tau_\alpha; \qquad P^{\alpha\epsilon} = \tau_\alpha^* \delta_\epsilon^* P^0 \delta_\epsilon \tau_\alpha. \tag{13}$$

If $\epsilon_0+\epsilon_1 < \alpha_1-\alpha_0$, then the operators $P^{\alpha_0\epsilon_0}$ and $P_{\alpha_1\epsilon_1}$ commute. In that case the following operator is an orthogonal projection:

$$P_{(\alpha_0,\alpha_1)} = P^{\alpha_0\epsilon_0}P_{\alpha_1\epsilon_1} \tag{14}$$

This projection maps smooth functions on the line into smooth functions supported in $[\alpha_0-\epsilon_0,\ \alpha_1+\epsilon_1]$. It may also be factored using the translated and dilated folding and unfolding operators. To do so we conjugate by dilation and translation to obtain a family of folding (respectively unfolding) operators indexed by the triple $(r, \alpha, \epsilon)$:

$$U(r,\alpha,\epsilon) = \tau_\alpha^*\delta_\epsilon^*U(r)\delta_\epsilon\tau_\alpha; \qquad U^*(r,\alpha,\epsilon) = \tau_\alpha^*\delta_\epsilon^*U^*(r)\delta_\epsilon\tau_\alpha. \tag{15}$$

For future reference, we expand the formulas for $U(r,\alpha,\epsilon)f$ and $U^*(r,\alpha,\epsilon)f$ and write them explicitly:

$$U\left(r,\alpha,\epsilon\right)f(t) = \begin{cases} r(\frac{t-\alpha}{\epsilon})f(t) + r(\frac{\alpha-t}{\epsilon})f(2\alpha-t), & \text{if } \alpha < t < \alpha+\epsilon, \\ \overline{r(\frac{\alpha-t}{\epsilon})}f(t) - \overline{r(\frac{t-\alpha}{\epsilon})}f(2\alpha-t), & \text{if } \alpha-\epsilon < t < \alpha, \\ f(t), & \text{otherwise;} \end{cases} \tag{16}$$

$$U^*\left(r,\alpha,\epsilon\right)f(t) = \begin{cases} \overline{r(\frac{t-\alpha}{\epsilon})}f(t) - r(\frac{\alpha-t}{\epsilon})f(2\alpha-t), & \text{if } \alpha < t < \alpha+\epsilon, \\ r(\frac{\alpha-t}{\epsilon})f(t) + \overline{r(\frac{t-\alpha}{\epsilon})}f(2\alpha-t), & \text{if } \alpha-\epsilon < t < \alpha, \\ f(t), & \text{otherwise.} \end{cases} \tag{17}$$

Where convenient we will write $U_0$ for $U(r_0,\alpha_0,\epsilon_0)$, and so on. We note that if the intervals $(\alpha_0-\epsilon_0,\ \alpha_0+\epsilon_0)$ and $(\alpha_1-\epsilon_1,\ \alpha_1+\epsilon_1)$ are disjoint, then the operators $U_0$, $U_1$, $U_0^*$ and $U_1^*$ all commute. In this case we will say that the pairs $(\alpha_0,\epsilon_0)$ and $(\alpha_1,\epsilon_1)$ are *consistent*. For consistent $(\alpha_0,\epsilon_0)$ and $(\alpha_1,\epsilon_1)$ the projection $P$ factors as follows:

$$P_{(\alpha_0,\alpha_1)} = U_0^*U_1^*\chi_{(\alpha_0,\alpha_1)}U_1U_0 \tag{18}$$

In the sequel we will make frequent use of these consistency hypotheses on the parameters $r, \alpha, \epsilon$ which we group together below for convenience:

*Definition.* We say that the *consistency conditions* hold for the triplets $(r_0,\alpha_0,\epsilon_0)$ and $(r_1,\alpha_1,\epsilon_1)$ if

(1) $r_0$ and $r_1$ belong to $C^d(\mathbf{R})$ for some $0 \le d \le \infty$ and satisfy Eq.(1);
(2) $\epsilon_0$ and $\epsilon_1$ are positive;
(3) The pairs $(\alpha_0,\epsilon_0)$ and $(\alpha_1,\epsilon_1)$ are consistent, i.e., the intervals $(\alpha_0-\epsilon_0,\ \alpha_0+\epsilon_0)$ and $(\alpha_1-\epsilon_1,\ \alpha_1+\epsilon_1)$ are disjoint.

## 3. Adjacent Compatible Intervals

The development of the Coifman–Malvar–Meyer bases in [AWW] defines *adjacent compatible intervals* to be the intervals $I = (\alpha_0,\alpha_1)$ and $J = (\alpha_1,\alpha_2)$ corresponding to mutually consistent pairs $(\alpha_i,\epsilon_i)$, $i = 0,1,2$, with associated smooth cutoffs $r_i$, $i = 0,1,2$ satisfying Eq.(1). We can give a simpler proof of one of the lemmas which appears in that paper:

**Lemma 3.** *If $I$ and $J$ are adjacent compatible intervals, then $P_I + P_J = P_{I\cup J}$ and $P_I P_J = P_J P_I = 0$.*

*Proof.* The operators $U_0$, $U_1$, $U_2$, $U_0^*$, $U_1^*$ and $U_2^*$ all commute because of the consistency condition. Furthermore, $U_0$ and $U_0^*$ commute with $\chi_J$ and $U_2$ and $U_2^*$ commute with $\chi_I$. Thus:

$$\begin{aligned} P_I + P_J &= U_0^* U_1^* \chi_I U_1 U_0 + U_1^* U_2^* \chi_J U_2 U_1 \\ &= U_1^* \left[ U_0^* \chi_I U_0 + U_2^* \chi_J U_2 \right] U_1 \\ &= U_0^* U_2^* U_1^* \left[ \chi_I + \chi_J \right] U_1 U_2 U_0 \end{aligned} \tag{19}$$

We note that $U_1$ and $U_1^*$ commute past $[\chi_I + \chi_J] = \chi_{I\cup J}$ and cancel. This shows that $P_I + P_J = P_{I\cup J}$.

Similarly, after interchanging various commuting operators we obtain

$$P_J P_I = P_I P_J = U_0^* U_1^* \chi_I U_1 U_0 U_1^* U_2^* \chi_J U_2 U_1 = U_0^* U_1^* U_2^* \chi_I \chi_J U_0 U_2 U_1 = 0. \quad \square \tag{20}$$

The factored construction for $P_I$ not only simplifies the proof of Lemma 3, it also points the way to a natural generalization. There is no reason to require that the map $P_I$ be a projection if what we want to do is to transform one orthonormal basis into another.

**Lemma 4.** *Write $I = (\alpha_0, \alpha_1)$. Suppose that $(r_0, \alpha_0, \epsilon_0), (r_1, \alpha_1, \epsilon_1)$ and $(r_2, \alpha_0, \epsilon_2), (r_3, \alpha_1, \epsilon_3)$ satisfy the consistency conditions. Define $U_0 = U(r_0, \alpha_0, \epsilon_0)$, $U_1 = U(r_1, \alpha_1, \epsilon_1)$, $V_0^* = U^*(r_2, \alpha_0, \epsilon_2)$ and $V_1^* = U^*(r_3, \alpha_1, \epsilon_3)$. If $f \in C^d(\mathbf{R})$, then $P_I f \stackrel{\text{def}}{=} V_0^* V_1^* \chi_I U_1 U_0 f$ has a unique continuous extension in $C^d(\mathbf{R})$ which is supported in the interval $[\alpha_0 - \epsilon_2,\ \alpha_1 + \epsilon_3]$. Furthermore, $P_I$ is a unitary isomorphism between $P_I^* L^2(\mathbf{R})$ and $P_I L^2(\mathbf{R})$, where $P_I^* \stackrel{\text{def}}{=} U_0^* U_1^* \chi_I V_2 V_1$.*

*Proof.* The smoothness and support properties of $P_I f$ follow from Lemma 1, since $\chi_I U_1 U_0 f$ satisfies Eq.(6) at $t = \alpha_0$ and $t = \alpha_1$ and has support in $I$.

Since $U_0$ and $U_1$ are unitary isomorphisms we may write $P_I L^2(\mathbf{R}) \cong V_0^* V_1^* \chi_I L^2(\mathbf{R})$. Similarly we may write $P_I^* L^2(\mathbf{R}) \cong U_0^* U_1^* \chi_I L^2(\mathbf{R})$, which shows that $P_I$ is an isomorphism between $P_I^* L^2(\mathbf{R})$ and $P_I L^2(\mathbf{R})$, with the inverse map being the adjoint $P_I^*$. $\square$

Notice that this proof shows $P_I L^2(\mathbf{R})$ to be unitarily isomorphic to $L^2(I)$, with the isomorphism given by $V_0 V_1$. This isomorphism replaces the traditional windowing of periodic bases. We remark that $P_I$ will be a projection if and only if $V_0^* U_0 = U_0 V_0^* = Id$ and $V_1^* U_1 = U_1 V_1^* = Id$. Since these are all unitary, it is equivalent that $U_0 = V_1$ and $U_1 = V_2$. If we use a single $\epsilon$ and a single $r$, then the interval $I = (\alpha_0, \alpha_1)$ and its adjacent translate by $|I| = \alpha_1 - \alpha_0$ are compatible. The main consequence of this observation is that $P_I$ composed with $|I|$-periodization is still a unitary isomorphism. This will be stated more precisely below.

## 4. Periodization

Our goal is to expand smooth functions in orthonormal bases of smooth compactly supported functions arising from arbitrary periodic bases. To achieve this goal we must first define the $\lambda$-*periodization* $\Omega_\lambda f$ of a function $f = f(t)$ by the usual formula:

$$\Omega_\lambda f(t) \stackrel{\text{def}}{=} \sum_{k\in\mathbf{Z}} f\left(t + k\lambda\right) = \sum_{k\in\mathbf{Z}} \tau_{k\lambda} f(t). \tag{21}$$

If $f$ belongs to $L^2(\mathbf{R})$ and is compactly supported, then $\Omega_\lambda f$ belongs to $L^2_{loc}(\mathbf{R})$ and is periodic of period $\lambda$. If in addition $f$ belongs to $C^d(\mathbf{R})$, then $\Omega_\lambda f$ also belongs to $C^d(\mathbf{R})$.

We can now define a "periodized" version of the folding and unfolding operators. The periodization is hidden in the definition of the following operators:

$$W\left(r,(\alpha_0,\alpha_1),\epsilon\right)f(t)=\begin{cases} r(\frac{t-\alpha_0}{\epsilon})f(t)+r(\frac{\alpha_0-t}{\epsilon})f(\alpha_0+\alpha_1-t), & \text{if } \alpha_0<t\le\alpha_0+\epsilon,\\ \overline{r(\frac{\alpha_1-t}{\epsilon})}f(t)-\overline{r(\frac{t-\alpha_1}{\epsilon})}f(\alpha_0+\alpha_1-t), & \text{if } \alpha_1-\epsilon\le t<\alpha_1,\\ f(t), & \text{otherwise;}\end{cases} \tag{22}$$

$$W^*\left(r,(\alpha_0,\alpha_1),\epsilon\right)f(t)=\begin{cases} \overline{r(\frac{t-\alpha_0}{\epsilon})}f(t)-r(\frac{\alpha_0-t}{\epsilon})f(\alpha_0+\alpha_1-t), & \text{if } \alpha_0<t\le\alpha_0+\epsilon,\\ r(\frac{\alpha_1-t}{\epsilon})f(t)+\overline{r(\frac{t-\alpha_1}{\epsilon})}f(\alpha_0+\alpha_1-t), & \text{if } \alpha_1-\epsilon\le t<\alpha_1,\\ f(t), & \text{otherwise.}\end{cases} \tag{23}$$

For these to be well defined, we must assume that $(r,\alpha_0,\epsilon)$ and $(r,\alpha_1,\epsilon)$ satisfy the consistency conditions.

Using $\Omega$, we can write the relationship between $W$ and $U$:

**Lemma 5.** *Suppose that $(r,\alpha_0,\epsilon)$ and $(r,\alpha_1,\epsilon)$ satisfy the consistency conditions. Then for all $t\in I=(\alpha_0,\alpha_1)$, we have:*

$$\begin{aligned} W(r,I,\epsilon)\,f(t)&=U(r,\alpha_0,\epsilon)U(r,\alpha_1,\epsilon)\Omega_{|I|}\chi_I\,f(t);\\ W^*(r,I,\epsilon)\,f(t)&=U^*(r,\alpha_0,\epsilon)U^*(r,\alpha_1,\epsilon)\Omega_{|I|}\chi_I\,f(t).\end{aligned} \tag{24}$$

*Proof.* We observe that for the periodic function $\tilde f=\Omega_{|I|}\chi_I f$ of period $|I|=\alpha_1-\alpha_0$ we have the following identity:

$$\tilde f(\alpha_0+\alpha_1-t)=\tilde f(2\alpha_0-t)=\tilde f(2\alpha_1-t). \tag{25}$$

Also, $f(t)=\tilde f(t)$ for all $t\in I$. Using these facts and Eq.(16), the formula for $W_I f$ inside $I$ becomes the following:

$$\begin{aligned} W\left(r,I,\epsilon\right)f(t)&=\begin{cases} r(\frac{t-\alpha_0}{\epsilon})\tilde f(t)+r(\frac{\alpha_0-t}{\epsilon})\tilde f(2\alpha_0-t), & \text{if } \alpha_0<t\le\alpha_0+\epsilon,\\ \overline{r(\frac{\alpha_1-t}{\epsilon})}\tilde f(t)-\overline{r(\frac{t-\alpha_1}{\epsilon})}\tilde f(2\alpha_1-t), & \text{if } \alpha_1-\epsilon\le t<\alpha_1,\\ f(t), & \text{otherwise;}\end{cases}\\ &=\begin{cases} U(r,\alpha_0,\epsilon)\tilde f(t), & \text{if } \alpha_0<t\le\alpha_0+\epsilon,\\ U(r,\alpha_1,\epsilon)\tilde f(t), & \text{if } \alpha_1-\epsilon\le t<\alpha_1.\\ f(t), & \text{otherwise.}\end{cases}\end{aligned} \tag{26}$$

The result follows immediately. □

When there is no possibility of confusion, we will write $W_I$ for $W(r,I,\epsilon)$ where $I=(\alpha_0,\alpha_1)$, suppressing the $r$ and $\epsilon$. We observe that $W_I$ and $W_I^*$ are unitary isomorphisms of $L^2(\mathbf{R})$ (i.e., $W_I^*W_I=W_IW_I^*=Id$) because $|r(t)|^2+|r(-t)|^2=1$ for all $t$. Also, if $t<\alpha_0$ or $t>\alpha_1$, then $W_If(t)=f(t)$ and $W_I^*f(t)=f(t)$. Thus if $I$ and $J$ are disjoint intervals, the operators $W_I$, $W_I^*$, $W_J$, and $W_J^*$ all commute. We also note that if $f$ is smooth in the interval $I$, then $W_If$ is also smooth there. Furthermore, $W_If$ satisfies the same boundary conditions at $\alpha_0+$ and $\alpha_1-$ as $U_0U_1f$:

**Lemma 6.** *Suppose that the consistency conditions hold for* $(r, \alpha_0, \epsilon)$ *and* $(r, \alpha_1, \epsilon)$*, and write* $I = (\alpha_0, \alpha_1)$*. If* $f \in C^d(\mathbf{R})$ *is* $|I|$*-periodic, then* $W_I f$ *belongs to* $C^d(\mathbf{R} \setminus \{\alpha_0, \alpha_1\})$*, has limits* $[W_I f]^{(n)}(\alpha_0+)$ *and* $[W_I f]^{(n)}(\alpha_1-)$ *for all* $0 \le n \le d$*, and satisfies the following conditions:*

$$\text{(27)} \qquad \begin{aligned} \lim_{t\to\alpha_0+} [W_I f]^{(n)}(t) = 0, &\qquad \text{if } n \text{ is odd;} \\ \lim_{t\to\alpha_1-} [W_I f]^{(n)}(t) = 0, &\qquad \text{if } n \text{ is even.} \end{aligned}$$

*Conversely, if* $f$ *belongs to* $C^d(I)$ *with limits* $f^{(n)}(\alpha_0+)$ *and* $f^{(n)}(\alpha_1-)$ *for all* $0 \le n \le d$ *which satisfy*

$$\text{(28)} \qquad \begin{aligned} \lim_{t\to\alpha_0+} f^{(n)}(t) = 0, &\qquad \text{if } n \text{ is odd;} \\ \lim_{t\to\alpha_1-} f^{(n)}(t) = 0, &\qquad \text{if } n \text{ is even,} \end{aligned}$$

*then* $W_I^* f$ *satisfies the equation*

$$\text{(29)} \qquad \lim_{t\to\alpha_0+} [W_I^* f]^{(n)}(t) = \lim_{t\to\alpha_1-} [W_I^* f]^{(n)}(t), \qquad \text{for all } 0 \le n \le d.$$

*Thus* $W_I^* f$ *has a continous periodic extension in* $C^d(\mathbf{R})$*.*

*Proof.* If $f$ is $|I|$-periodic, then $\Omega_{|I|}\chi_I f = f$. Thus Eq.(27) follows from Lemma 5 and an application of Lemma 1 at $\alpha_0+$ and $\alpha_1-$.

Given Eq.(28) we deduce that $\tilde{f} = \Omega_{|I|}\chi_I f$ satisfies the conditions

$$\text{(30)} \qquad \begin{aligned} \tilde{f}^{(n)}(\alpha_0+) = \tilde{f}^{(n)}(\alpha_1+) = 0, &\qquad \text{if } n \text{ is odd;} \\ \tilde{f}^{(n)}(\alpha_0-) = \tilde{f}^{(n)}(\alpha_1-) = 0, &\qquad \text{if } n \text{ is even.} \end{aligned}$$

We can evaluate the one-sided limits of Eq.(29) by using Lemma 5:

$$\text{(31)} \qquad \begin{aligned} \left[W^*(r, I, \epsilon) f\right]^{(n)}(\alpha_0+) = \left[U_0^* \tilde{f}\right]^{(n)}(\alpha_0+), &\qquad \text{for all } 0 \le n \le d; \\ \left[W^*(r, I, \epsilon) f\right]^{(n)}(\alpha_1-) = \left[U_1^* \tilde{f}\right]^{(n)}(\alpha_1-), &\qquad \text{for all } 0 \le n \le d. \end{aligned}$$

But $U_1^* \tilde{f} = \tau_{|I|}^* U_0^* \tau_{|I|} \tilde{f} = \tau_{|I|}^* U_0^* \tilde{f}$, since $\tilde{f}$ is $|I|$-periodic, so that $\left[U_0^* \tilde{f}\right]^{(n)}(\alpha_0+) = \left[U_1^* \tilde{f}\right]^{(n)}(\alpha_1+)$ and $\left[U_0^* \tilde{f}\right]^{(n)}(\alpha_0-) = \left[U_1^* \tilde{f}\right]^{(n)}(\alpha_1-)$. Finally, the converse of Lemma 1 applied at $\alpha_0$ (or just as well at $\alpha_1$) implies that $\left[U_0^* \tilde{f}\right]^{(n)}(\alpha_0+) = \left[U_0^* \tilde{f}\right]^{(n)}(\alpha_0-)$ for all $0 \le n \le d$, from which follows Eq.(29). □

## 5. Orthonormal Bases

**Lemma 7.** *Suppose that* $(r, \alpha_0, \epsilon), (r, \alpha_1, \epsilon)$ *and* $(r_0, \alpha_0, \epsilon_0), (r_1, \alpha_1, \epsilon_1)$ *satisfy the consistency conditions, and write* $I = (\alpha_0, \alpha_1)$ *and* $W_I = W(r, I, \epsilon)$*. If* $\{e_j\}_{j\in\mathbf{Z}}$ *is a collection of* $|I|$*-periodic functions which form an orthonormal basis for* $L^2(I)$ *when restricted to* $I$*, then* $E_0 = \{U_0^* U_1^* \chi_I W_I\, e_j\}_{j\in\mathbf{Z}}$ *is an orthonormal basis of* $U_0^* U_1^* L^2(\mathbf{R})$*. In addition, if* $\{e_j\}_{j\in\mathbf{Z}} \subset C^d(\mathbf{R})$*, then* $E_0 \subset C_0^d(\mathbf{R})$*.*

*Proof.* The functions $\{\chi_I W_I e_j\}_{j\in\mathbf{Z}}$ form an orthonormal basis of $L^2(I)$, since $W_I$ is unitary. Then $E_0$ is an orthonormal basis of $U_0^* U_1^* \chi_I L^2(\mathbf{R})$ since $U_0^* U_1^*$ is unitary on $L^2(\mathbf{R})$.

Lemma 6 implies that $\chi_I W_I e_j$ satisfies Eq.(6) at $\alpha_0$ and $\alpha_1$. Then the converse of Lemma 1 implies that each function in $E_0$ belongs to $C_0^d(\mathbf{R})$, in fact with support in the interval $[\alpha_0 - \epsilon_0,\, \alpha_1 + \epsilon_1]$. □

We can define a segmentation of the line into arbitrary windows with varying overlaps. Fix a sequence $\{(r_k, \alpha_k, \epsilon_k) : k \in \mathbf{Z}; \epsilon_k, a_k \in \mathbf{R}; \epsilon_k > 0\}$ such that adjacent triplets satisfy the consistency conditions. We also require that $a_k < a_{k+1}$ for all $k \in \mathbf{Z}$, and that

$$\mathbf{R} = \bigcup_{k\in\mathbf{Z}} [a_k, a_{k+1}) \stackrel{\text{def}}{=} \bigcup_{k\in\mathbf{Z}} I_k \tag{32}$$

Note that $U_k = U(r_k, \alpha_k, \epsilon_k)$ and $U_l = U(r_l, \alpha_l, \epsilon_l)$ commute for all $k, l \in \mathbf{Z}$. We likewise define $U_k^* = U^*(r_k, \alpha_k, \epsilon_k)$. The first main point of this paper is that from arbitrary smooth *periodic* orthonormal bases we can construct smooth *compactly supported* orthonormal bases for $L^2(\mathbf{R})$. This is a consequence of our previous lemmas, which combine to give the following:

**Theorem 8.** *Suppose that*

1. $\mathbf{R}$ *is decomposed as in Eq.*(32)*;*
2. *for each $k \in \mathbf{Z}$ the triplets $(r_k, \alpha_k, \epsilon_k)$ and $(r_{k+1}, \alpha_{k+1}, \epsilon_{k+1})$ satisfy the consistency conditions;*
3. *for each $k \in \mathbf{Z}$ the family of $|I_k|$-periodic functions $\{e_{k,j} : j \in \mathbf{Z}\}$ has the property that when restricted to $I_k$ it forms an orthonormal basis of $L^2(I_k)$;*
4. *for each $k \in \mathbf{Z}$ the pair $(\tilde{r}_k, \alpha_k, \tilde{\epsilon}_k), (\tilde{r}_k, \alpha_{k+1}, \tilde{\epsilon}_k)$ satisfy the consistency conditions.*

*We define $U_k^* = U^*(r_k, \alpha_k, \epsilon_k)$ and $W_k = W(\tilde{r}_k, I_k, \tilde{\epsilon}_k)$. Then the collection*

$$E = \{U_k^* U_{k+1}^* \chi_{I_k} W_k \, e_{k,j} : j, k \in \mathbf{Z}\} \tag{33}$$

*is an orthonormal basis for $L^2(\mathbf{R})$ consisting of functions of compact support. If in addition all the functions $e_{k,j}$ and $r_k$, $k, j \in \mathbf{Z}$, belong to $C^d(\mathbf{R})$ for some $0 \le d \le \infty$, then the functions in $E$ belong to $C_0^d(\mathbf{R})$.*

*Proof.* Since adjacent intervals $I_k, I_{k+1}$ are compatible for all $k \in \mathbf{Z}$, Lemma 3 gives us the decomposition $L^2(\mathbf{R}) = \bigoplus_{k\in\mathbf{Z}} P_{I_k} L^2(\mathbf{R})$. By Lemma 7, each of the spaces $P_I L^2(\mathbf{R})$ has an orthonormal basis $E_k = \{U_k^* U_{k+1}^* \chi_{I_k} W_k \, e_{k,j} : j \in \mathbf{Z}\}$. Putting these bases together into $E = \bigcup_{k\in\mathbf{Z}} E_k$ yields the result. □

In practice it is often better to transform a smooth function into a smooth periodic function and then expand it in a periodic basis, rather than expand a smooth function in the basis $E$ described above. This is because well-tested computer programs exist for the first algorithm but not the second. The second main result is just the adjoint of Theorem 8:

**Theorem 9.** *Suppose that the triplets $(r_0, \alpha_0, \epsilon_0), (r_1, \alpha_1, \epsilon_1)$ and $(r, \alpha_0, \epsilon), (r, \alpha_1, \epsilon)$ satisfy the consistency conditions. Write $I = (\alpha_0, \alpha_1)$, $U_0 = U(r_0, \alpha_0, \epsilon_0)$, $U_1 = U(r_1, \alpha_1, \epsilon_1)$, and $W_I^* = W^*(r, I, \epsilon)$. If $f$ belongs to $C^d(\mathbf{R})$, then $T_I f \stackrel{\text{def}}{=} W_I^* \chi_I U_0 U_1 \, f$ has an $I$-periodic extension which belongs to $C^d(\mathbf{R})$. Also, $T_I$ is a unitary isomorphism from $U_0^* U_1^* \chi_I L^2(\mathbf{R})$ to $L^2(I)$.*

*Proof.* Since $\chi_I U_0 U_1 \, f$ satisfies Eq.(28), the converse of Lemma 6 implies that

$$\lim_{t\to\alpha_0+} [W_I^* \chi_I U_0 U_1 f]^{(n)}(t) = \lim_{t\to\alpha_1-} [W_I^* \chi_I U_0 U_1 f]^{(n)}(t),$$

for all $0 \le n \le d$. Hence $W_I^* \chi_I U_0 U_1 f$ has a unique continous periodic extension in $C^d(\mathbf{R})$.

For the second part, we note that $U_0 U_1$ is a unitary isomorphism from $U_0^* U_1^* \chi_I L^2(\mathbf{R})$ to $\chi_I L^2(\mathbf{R}) \cong L^2(I)$, and $W_I^* \chi_I$ is a unitary automorphism on $\chi_I L^2(\mathbf{R})$. □

## 6. Example bases

We consider some examples of the action of a specific folding and unfolding operator. The Mathematica™ program which generated the graphs below is available in electronic form by anonymous ftp [FD]. Let $r$ be defined as in Eq.(2), let $\alpha_0 = 0$, $\alpha_1 = 4$, $I = (0, 4)$, and $\epsilon = \epsilon_0 = \epsilon_1 = 1$. From the function $e_4(t) = e^{2\pi i t}$ we get the following basis function for $L^2(I)$:

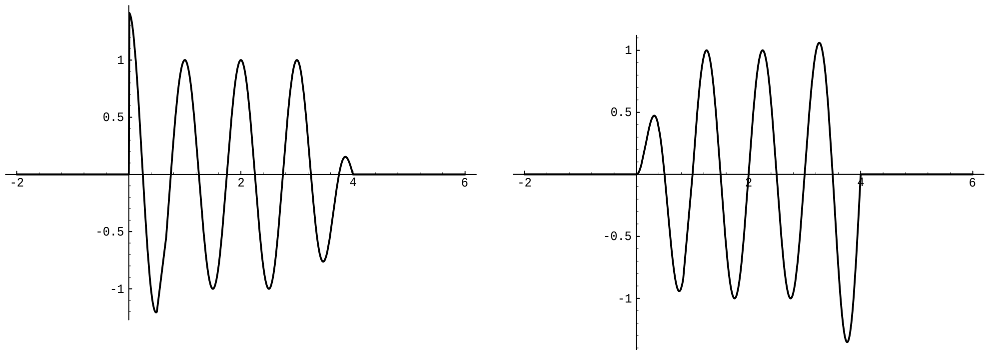


**Figure 3.**

**Real and imaginary parts of $\chi_I W_I e_4$.**

When these are unfolded, we obtain the following basis function for $P_I L^2(\mathbf{R})$:

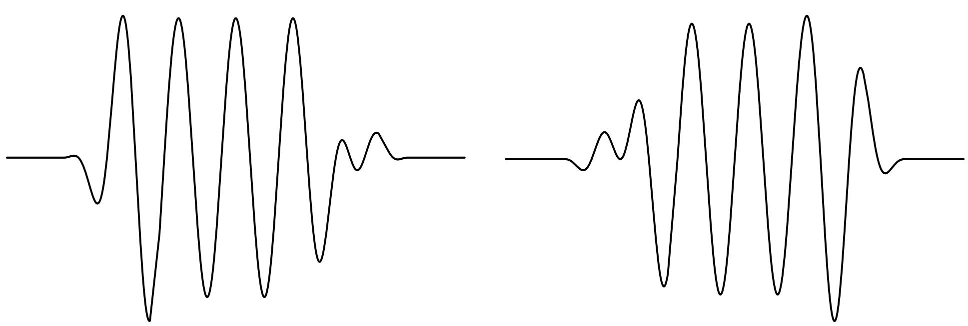

**Figure 4.**

**Real and imaginary parts of $U_0^* U_1^* \chi_I W_I e_4$.**

Conversely, computing the frequency 4 member of the Fourier series of the function $W_I^* \chi_I U_0 U_1 f$ is equivalent to finding the inner product of $f$ with $U_0^* U_1^* \chi_I W_I e_4$.

## 7. Historical Note

The projection operators $P^0$ and $P_0$ were first described to the author by R. R. Coifman in 1985, in a discussion about some work of Y. Meyer. Originally they were used to approximate the Hilbert transform with $H_\epsilon$ defined by $\widehat{(H_\epsilon f)} = P_{0\epsilon}\hat{f} - P^{0\epsilon}\hat{f}$, a better-behaved operator which retains the algebraic properties of the original. Their usefulness to signal processing became most apparent during the author's implementation of the Malvar transform in a package of adapted waveform analysis computer programs [AWA].

## 8. Appendix: Source code

The following is an implementation of the operator $U$ described above. It folds an array in place:

```
/* fold()
 *
 * Given an array, smoothly fold the odd part into the left half and the even
 * part into the right half.  This is implemented as a transformation in place
 * of two arrays:
 *             –the left half–                    –the right half–
 *    ominus[-reach] . . . ominus[-1]       oplus[0] . . . oplus[reach-1]
 *
 * Here reach is a positive integer.  This indexing is chosen so that ‘oplus’
 *  and ‘ominus’ are typically identical pointers to the first element of a
 *  block of the given array.  The function then folds the leading edge of the
 *  block into the trailing edge of the previous block.  However, the array
 *  locations ‘ominus[-reach]...ominus[-1], oplus[0], ..., oplus[reach-1]’
 *  must not overlap.
 *
 * The formulas for the transformation are:
 *  oplus[k] = rise[-k-1]*ominus[-k-1] + rise[k]*oplus[k],    k= 0,...,reach-1
 *  ominus[k] = rise[-k-1]*ominus[k] - rise[k]*oplus[-k-1],    k= -reach,...,-1
 *
 * Temporary variables are used to allow transformation in place.
 *
 * Calling sequence:
 *         fold( ominus, oplus, rise, reach )
 *
 * Input:
 *         (real *)ominus                Values ominus[-1]...ominus[-reach] must be
 *                                 allocated and defined.
 *
 *          (real *)oplus                Values oplus[0]...oplus[reach-1] must be
 *                                 allocated and defined.
 *
 *         (const real *)rise        This array must contain a valid rise
 *                                 function’s ”left half”, increasing from
 *                                 ‘rise[-reach]’ to ‘rise[reach-1]’.
 *
 *         (int)reach                  This positive integer determines the size of
 *                                 the rise and the range of the folding.
 *
 * Output:
```

```
 *   Array values oplus[0]...oplus[reach-1] and ominus[-1]...ominus[-reach]
 *     are replaced with orthogonal linear combinations.
 */
void
fold(
      real *ominus,          /* Negative (left) side: -reach ≤ x < 0. */
      real *oplus,           /* Positive (right) side: 0 ≤ x < reach. */
      const real *rise,      /* Rise increases on -reach ≤ x < reach. */
      int   reach)           /* Positive integer. */
{
  int k;
  real templus;
  const real *rneg;

  rneg = rise;
  for( k=0; k<reach; ++k)
    {
      templus   = (*rise)*(*oplus) + (*--rneg)*(*--ominus);
      *ominus   = (*rise++)*(*ominus) - (*rneg)*(*oplus);
      *oplus++  = templus;
    }
  return;
}
```

The following is an implementation of the operator $U^*$ described above. It unfolds an array in place:

```
/* unfold()
 *
 * Given an array, smoothly fold the even part into the left half and the odd
 * part into the right half.  This is implemented as a transformation in place
 * of two arrays:
 *            –the left half–                    –the right half–
 *    ominus[-reach] . . . ominus[-1]       oplus[0] . . . oplus[reach-1]
 *
 * Here reach is a positive integer.  This indexing is chosen so that 'oplus'
 *  and 'ominus' are typically identical pointers to the first element of a
 *  block of the given array.  The function then folds the leading edge of the
 *  block into the trailing edge of the previous block.  However, the array
 *  locations 'ominus[-reach]...ominus[-1], oplus[0], ..., oplus[reach-1]'
 *  must not overlap.
 *
 * The formulas for the transformation are:
 *  oplus[k] =  rise[k]*oplus[k] - rise[-k-1]*ominus[-k-1],    k= 0,...,reach-1
 *  ominus[k] = rise[-k-1]*ominus[k] + rise[k]*oplus[-k-1],    k= -reach,...,-1
```

```
 *
 * Temporary variables are used to allow transformation in place.
 *
 * Calling sequence:
 *       unfold( ominus, oplus, rise, reach )
 *
 * Input:
 *       (real *)ominus             Values ominus[-1]...ominus[-reach] must be
 *                                  allocated and defined.
 *
 *       (real *)oplus              Values oplus[0]...oplus[reach-1] must be
 *                                  allocated and defined.
 *
 *       (const real *)rise         This array must contain a valid bell
 *                                  function's "left half", increasing from
 *                                  'rise[-reach]' to 'rise[reach-1]'.
 *
 *       (int)reach                 This positive integer determines the size of
 *                                  the rise and the range of the folding.
 * Output:
 *   Array values oplus[0]...oplus[reach-1] and ominus[-1]...ominus[-reach]
 *     are replaced with orthogonal linear combinations.
 */
void
unfold(
      real *ominus,          /* Negative (left) side: -reach <= x < 0. */
      real *oplus,           /* Positive (right) side: 0 <= x < reach. */
      const real *rise,      /* Rise increases on -reach <= x < reach. */
      int   reach)           /* Positive integer. */
{
  int k;
  real templus;
  const real *rneg;

  rneg = rise;
  for( k=0; k<reach; ++k)
    {
      templus   = (*rise)*(*oplus) - (*--rneg)*(*--ominus);
      *ominus   = (*rise++)*(*ominus) + (*rneg)*(*oplus);
      *oplus++  = templus;
    }
  return;
}
```

## References

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