Time Localization Techniques for Wavelet Transforms

Mladen Victor Wickerhauser*
Department of Mathematics
Washington University in St. Louis
Missouri 63130 USA

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Abstract

We consider the following pair of problems related to orthonormal compactly supported wavelet expansions: (1) Given a wavelet coefficient with its nominal scale and position indices, find the precise location of the transient signal feature which produced it; (2) Given two collections of wavelet coefficients, determine whether they arise from a periodic signal and its translate, and if so find the translation which maps one into the other. Both problems may be solved by traditional means after inverting the wavelet transform, but we propose two alternative algorithms which rely solely on the wavelet coefficients themselves.

Keywords: Discrete orthogonal wavelets, translation invariance, QMF, FIR, linear phase filters.

1 Introduction

Continuous wavelet decompositions of functions [7] have now been used for more than a decade to extract the locations and properties of transient features of time-varying, nonstationary signals. Basic algorithms, such as retaining only the largest wavelet components and determining the time location of their basis elements [10], produce excellent results in cases such as isolating discontinuities or frequency transitions in music and speech. More sophisticated algorithms can locate and model transient phenomena very precisely, for instance to remove certain dominant but uninteresting background features like solvent absorbances in NMR spectrograms [8], or to replace a textured image by a textureless cartoon [12]. However, the computational time and space costs of the continuous wavelet transform—it produces a two-dimensional data set from a one-dimensional input—prevent the use of such methods in high-speed or "real-time" applications.

Discrete, compactly supported orthonormal wavelet bases, introduced by Daubechies [3], would be a formidable replacement tool for these transient signal processing and feature detection problems because of their much lower computational complexity. They provide a real-valued transformation which preserves both dimension and rank, i.e., N-point one-dimensional real inputs produce N-point one-dimensional real outputs. There are a number of problems, however, caused by artifacts associated to the dyadic subsampling or "decimation" used in the discrete wavelet transform.

The support of compactly supported orthonormal wavelets grows as more regularity is required, and extra regularity is often desirable for the representation of smooth or highly correlated signals. Most of the mass of a unit scale compactly supported wavelet lies over an interval of unit width, though the actual support is equal to the filter length, typically 10 or 20 units. We first consider the problem of locating the *center of energy* of a wavelet within this support, given its scale and position indices. This can be done exactly for symmetric and antisymmetric wavelets, but the best we can do in the general case is to locate the center up to a signal dependent error which is bounded by the wavelet's *deviation from linear phase*, or deviation from symmetry or antisymmetry. We compute a quantity to measure this deviation somewhat differently from Daubechies [4]. Our goal is to associate two numbers to each wavelet which can be used to correct the nominal center of energy and locate it more precisely.

It is well known that the discrete wavelet transform is very sensitive to small translations of its input. A signal consisting of a single basis wavelet which has been shifted slightly from its grid, for example, can have a discrete wavelet transform in which all the coefficients have nearly the same amplitude. But when shifted to its proper

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location, the one-wavelet signal will be easily recognized by its single nonzero coefficient. We will describe a fast algorithm, first introduced by Beylkin [1], which computes for us the best circulant shift to apply to a periodic signal before performing a discrete orthonormal wavelet transform, so as to obtain the most peaked sequence of wavelet coefficients. Such an algorithm would detect that a signal consists of a single wavelet. It also serves to compare, in wavelet coefficients, two signals differing only by a shift.

2 Localizing Transients Given Wavelet Coefficients

We follow the notation conventions and terminology used in [16]. A square-integrable function u defines two probability density functions: $x \mapsto |u(x)|^2/\|u\|^2$ and $\xi \mapsto |\hat{u}(\xi)|^2/\|\hat{u}\|^2$. It is not possible for both of these densities to be arbitrarily concentrated, as we shall see from the inequalities below.

2.1 Heisenberg's Inequality

Suppose that $u = u(x)$ belongs to the Schwartz class $\mathcal{S}$. Then $x\frac{d}{dx}|u(x)|^2 = x\,[u(x)\bar{u}'(x) + \bar{u}(x)u'(x)]$ is integrable and tends to 0 as $|x| \to \infty$. We can therefore integrate by parts to get the following formula:

$$\int_{\mathbf{R}} -x\frac{d}{dx}|u(x)|^2\,dx = \int_{\mathbf{R}} |u(x)|^2\,dx = \|u\|^2. \tag{1}$$

But also, we have the following consequences of the Cauchy–Schwarz inequality ($|\langle f,g\rangle| \le \|f\|\,\|g\|$) and the triangle inequality ($\|x-z\| \le \|x-y\| + \|y-z\|$):

$$\left|\int_{\mathbf{R}} -x\frac{d}{dx}|u(x)|^2\,dx\right| \le 2\int_{\mathbf{R}} |xu(x)u'(x)|\,dx \le 2\left(\int_{\mathbf{R}} |xu(x)|^2\,dx\right)^{1/2}\left(\int_{\mathbf{R}} |u'(x)|^2\,dx\right)^{1/2}.$$

Combining the last two inequalities gives $\|xu(x)\|\cdot\|u'(x)\| \ge \frac{1}{2}\|u(x)\|^2$. Now $\widehat{u'}(\xi) = 2\pi i\xi\hat{u}(\xi)$, and $\|\hat{v}\| = \|v\|$ by Plancherel's theorem, so we can rewrite the inequality as follows:

$$\frac{\|xu(x)\|}{\|u(x)\|}\cdot\frac{\|\xi\hat{u}(\xi)\|}{\|\hat{u}(\xi)\|} \ge \frac{1}{4\pi}.$$

Since the right-hand side is not changed by translation $u(x) \mapsto u(x-x_0)$ or modulation $\hat{u}(\xi) \mapsto \hat{u}(\xi-\xi_0)$, we have proved

$$\inf_{x_0}\left(\frac{\|(x-x_0)u(x)\|}{\|u(x)\|}\right)\cdot\inf_{\xi_0}\left(\frac{\|(\xi-\xi_0)\hat{u}(\xi)\|}{\|\hat{u}(\xi)\|}\right) \ge \frac{1}{4\pi}. \tag{2}$$

Equation 2 is called *Heisenberg's inequality.* We mention the usual names

$$\triangle\, x = \triangle x(u) \stackrel{\text{def}}{=} \inf_{x_0}\left(\frac{\|(x-x_0)u(x)\|}{\|u(x)\|}\right); \qquad \triangle\xi = \triangle\xi(u) \stackrel{\text{def}}{=} \inf_{\xi_0}\left(\frac{\|(\xi-\xi_0)\hat{u}(\xi)\|}{\|\hat{u}(\xi)\|}\right). \tag{3}$$

The quantities $\triangle x$ and $\triangle\xi$ are called the *uncertainties* in position and momentum respectively, and they provide an inverse measure of how well u and $\hat{u}$ are localized. Then Heisenberg's inequality assumes the guise of the *uncertainty principle*:

$$\triangle\, x\cdot\triangle\xi \ge \frac{1}{4\pi}. \tag{4}$$

It is not hard to show that the infima in 3 are attained at the points x_0 and ξ_0 defined by the following expressions:

$$x_0 = x_0(u) = \frac{1}{\|u\|^2}\int_{\mathbf{R}} x|u(x)|^2\,dx; \qquad \xi_0 = \xi_0(u) = \frac{1}{\|\hat{u}\|^2}\int_{\mathbf{R}} \xi|\hat{u}(\xi)|^2\,d\xi. \tag{5}$$

The Dirac mass $\delta(x-x_0)$ is perfectly localized at position x_0, with zero position uncertainty, but both its frequency and frequency uncertainty are undefined. Likewise, the exponential $e^{2\pi i\xi_0 x}$ is perfectly localized in momentum (since its Fourier transform is $\delta(\xi-\xi_0)$), but both its position and position uncertainty are undefined. Equality is obtained in Equations 2 and 4 if we use the *Gaussian function* $u(x) = e^{-\pi x^2}$. It is possible to show, using the uniqueness theorem for solutions to linear ordinary differential equations, that the only functions which minimize Heisenberg's inequality are scaled, translated, and modulated versions of the Gaussian function.

If $\triangle x$ and $\triangle\xi$ are both finite, then the quantities x_0 and ξ_0 can be used to assign a nominal position and momentum to an imperfectly localized function.

2.2 Convolution

Given two sequences $u = \{u(n)\}_{n\in G}$ and $v = \{v(n)\}_{n\in G}$, their *convolution* is the sequence $u * v$ defined by

$$u * v(n) \stackrel{\text{def}}{=} \sum_{k\in G_n} u(k)v(n-k) = \sum_{k\in G_n} u(n-k)v(k); \qquad G_n \stackrel{\text{def}}{=} \{k \in G : n-k \in G\}. \tag{6}$$

This is defined for $n \in G$. We will consider two choices of index set: the complete set of integers, and the integers modulo some period $q > 0$.

2.2.1 Doubly Infinite Sequences

If the sequences u and v are defined at all the integers $G = \mathbf{Z}$, then the convoultion formula reduces to the infinite sum

$$u * v(x) = \sum_{y=-\infty}^{\infty} u(y)v(x-y).$$

Proposition 2.1 *If $u \in \ell^1(\mathbf{Z})$ and $v \in \ell^p(\mathbf{Z})$ for $1 \le p \le \infty$, then $u * v \in \ell^p$.* □

We can compute convolutions efficiently by multiplication of Fourier transforms:

Proposition 2.2 *If u and v are infinite sequences such that $\hat{u}$ and $\hat{v}$ exist a.e., then $\widehat{u * v}(\xi) = \hat{u}(\xi)\hat{v}(\xi)$ for almost every $\xi \in \mathbf{T}$.* □

Proposition 2.3 *If $u \in \ell^1(\mathbf{Z})$, then the map $v \mapsto u * v$ has operator norm $\max_{\xi\in\mathbf{T}} |\hat{u}(\xi)|$ as a map from $L^2(\mathbf{T})$ to $L^2(\mathbf{T})$.* □

The special case which will interest us the most is that of "finitely supported" sequences, *i.e.*, those for which $u(x) = 0$ except for finitely many integers x. Such sequences are obviously summable, and it is easy to show that the convolution of finitely supported sequences is also finitely supported. Furthermore, if u is finitely supported, then $\hat{u}$ is a trigonometric polynomial and we may use many powerful tools from classical analysis to study it.

So, let $u = u(x)$ and $v = v(x)$ be finitely supported sequences taking values at integers $x \in \mathbf{Z}$, with $u(x) = 0$ unless $a \le x \le b$ and $v(x) = 0$ unless $c \le x \le d$. We call $[a, b]$ and $[c, d]$ the *support intervals* $\operatorname{supp} u$ and $\operatorname{supp} v$, respectively, and $b - a$ and $d - c$ the *support widths* for the sequences u and v. Then $u * v(x) = 0$ unless there is some $y \in \mathbf{Z}$ for which $y \in [a, b]$ and $x - y \in [c, d]$, which requires that $c + a \le x \le d + b$. Hence $u * v$ is also finitely supported, with the width of its support growing to $(d + b) - (c + a) = (b - a) + (d - c)$, or the sum of the support widths of u and v. The convolution at x is a sum over $y \in [a, b] \cap [x - d, x - c]$.

2.2.2 Periodic Sequences

If $G = \mathbf{Z}/q\mathbf{Z}$ is the integers $\{0, 1, \ldots, q-1\}$ with addition modulo q, then the convolution integral becomes a finite sum:

$$u * v(x) = \sum_{y=0}^{q-1} u(y)v(x - y \bmod q).$$

Since all sequences in this case are finite, there is no question of summability. Convolution becomes multiplication via the discrete Fourier transform:

$$\hat{v}(k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v(j)e^{-2\pi ijk/N}, \qquad k = 0, 1, \ldots, N-1. \tag{7}$$

Proposition 2.4 *If u, v are q-periodic sequences, then $\widehat{u * v}(y) = \hat{u}(y)\hat{v}(y)$.* □

Thus we can compute the norm of discrete convolution operators:

Proposition 2.5 *The operator norm of the map $v \mapsto u * v$ from $\ell^2(\mathbf{Z}/q\mathbf{Z})$ to itself is $\max_{0\le y<q} |\hat{u}(y)|$.*

Proof: The maximum is achieved for the sequence $v(x) = \exp 2\pi ixy_0/q$, where y_0 is the maximum for $|\hat{u}|$, since then $\hat{v}(y) = \sqrt{q}\,\delta(y - y_0)$. □

Periodic convolution is the efficient way to apply a convolution operator to a periodic sequence. Suppose that $v \in \ell^\infty(\mathbf{Z})$ happens to be q-periodic, namely that $v(x+q) = v(x)$ for all $x \in \mathbf{Z}$. Then for $u \in \ell^1(\mathbf{Z})$ we can compute the convolution of u and v by decomposing $y = k + qn$:

$$u * v(x) = \sum_{y=-\infty}^{\infty} u(y)v(x-y) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{q-1} u(k+qn)v(x-k-qn) = \sum_{k=0}^{q-1} \left(\sum_{n=-\infty}^{\infty} u(k+qn) \right) v(x-k).$$

Now let us define the *q-periodization* u_q of $u \in \ell^1(\mathbf{Z})$ to be the q-periodic function

$$u_q(k) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} u(k+qn). \tag{8}$$

Thus starting with a single sequence u, we can get a family of convolution operators, one on $\mathbf{Z}/q\mathbf{Z}$ for each integer $q > 0$:

$$U_q : \ell^2(\mathbf{Z}/q\mathbf{Z}) \to \ell^2(\mathbf{Z}/q\mathbf{Z}); \qquad U_q v(x) = u_q * v(x) = \sum_{k=0}^{q-1} u_q(k)v(x-k). \tag{9}$$

In effect, we *preperiodize* the sequence u to any desired period q before applying the convolution operator.

2.2.3 Convolution as an Operator

The Fourier transform converts convolution into pointwise multiplication. We can use this result together with Plancherel's theorem to prove that convolution with integrable functions preserves square-integrability. Suppose that u is integrable and v is square-integrable. Then by Plancherel's theorem and the convolution theorem we have $\|u * v\| = \|\widehat{u * v}\| = \|\hat{u}\,\hat{v}\|$. This gives the estimate

$$\|u * v\| \leq \|\hat{u}\|_\infty \|\hat{v}\| = \|\hat{u}\|_\infty \|v\| \leq \|u\|_{L^1} \|v\|. \tag{10}$$

Convolution with integrable u is a bounded linear operator on L^2, and we will have occasion to estimate this bound with the following proposition:

Proposition 2.6 *If $u = u(x)$ is absolutely integrable on $\mathbf{R}$, then the convolution operator $v \mapsto u * v$ as a map from L^2 to L^2 has operator norm* $\sup\{|\hat{u}(\xi)| : \xi \in \mathbf{R}\}$.

Proof: By Equation 10, $\|u * v\| \leq \sup\{|\hat{u}(\xi)|\,\|v\| : \xi \in \mathbf{R}\}$. By the Riemann–Lebesgue lemma, $\hat{u}$ is bounded and continuous and $|\hat{u}(\xi)| \to 0$ as $|\xi| \to \infty$, so $\hat{u}$ achieves its maximum amplitude $\sup\{|\hat{u}(\xi)| : \xi \in \mathbf{R}\} < \infty$ at some point $\xi_* \in \mathbf{R}$. We may assume without loss that $\xi_* = 0$. To show that the operator norm inequality is sharp, let $\epsilon > 0$ be given and find $\delta > 0$ such that $|\xi - \xi_*| < \delta \Rightarrow |\hat{u}(\xi) - \hat{u}(\xi_*)| < \epsilon$. If we take $v(x) = \frac{\sin 2\pi\delta x}{\pi x}$, then $\hat{v}(\xi) = \mathbf{1}_{[-\delta,\delta]}(\xi)$, and $\|u * v\| = \|\hat{u}\hat{v}\| > (1-\epsilon)\,|\hat{u}(\xi_*)|\,\|\hat{v}\| = (1-\epsilon)\,|\hat{u}(\xi_*)|\,\|v\|$. □

2.3 Decimation and Shifts

Decimation by q can be regarded as the process of discarding all values of a sampled function except those indexed by a multiple of $q > 0$. We denote it by d_q, and we have

$$[d_q u](n) \stackrel{\text{def}}{=} u\,(qn)\,. \tag{11}$$

If $u = \{u(n) : n \in \mathbf{Z}\}$ is an infinite sequence, then the new infinite sequence $d_q u$ is just $\{u(qn) : n \in \mathbf{Z}\}$ or every q^{th} element of the original sequence.

If u is finitely supported and $\operatorname{supp} u = [a,b]$, then $d_q u$ is also finitely supported and $\operatorname{supp} d_q u = [a,b] \bigcap q\mathbf{Z}$. This set contains either $\left\lfloor \frac{|b-a|}{q} \right\rfloor$ or $\left\lfloor \frac{|b-a|}{q} \right\rfloor + 1$ elements.

If u is a periodic sequence of period p, then $d_q u$ has period $q/\gcd(p,q)$. Counting degrees of freedom, the number of q-decimated subsequences of a p-periodic sequence needed to reproduce it is exactly $\gcd(p,q)$. If $\gcd(p,q) = 1$, then decimation is just a permutation of the original sequence and there is no reason to perform it. Thus, in the typical case of $q = 2$ we will always assume that p is even.

The *translation* or *shift* operator τ_y is defined by

$$\tau_y u(x) = u(x-y). \tag{12}$$

Whatever properties u has at $x = 0$ the function $\tau_y u$ has at $x = y$. Observe that τ_0 is the identity operator. Translation invariance is a common property of formulas derived from physical models because the choice of "origin" 0 as in $u(0)$ for an infinite sequence is usually arbitrary. Any functional or measurement computed for u which does not depend on this choice of origin must give the same value for the sequence $\tau_y u$, regardless of y. For example, the energy $\|u\|^2$ in a sequence does not depend on the choice of origin:

$$\text{For all } y, \qquad \|u\|^2 = \|\tau_y u\|^2. \tag{13}$$

Such invariance can be used to algebraically simplify formulas for computing the measurement.

Translation and dilation do not commute in general, but there is an "intertwining" relation

$$\text{For all } x, y, p, \qquad \tau_y \sigma_p u(x) = \sigma_p \tau_{y/p} u(x). \tag{14}$$

Let t_y denote translation in the discrete case: $t_y u(n) \stackrel{\text{def}}{=} u(n-y)$. The intertwining relation then becomes $t_y d_p u = d_p t_{py} u$.

2.4 Quadrature Filters

We shall use the term *quadrature filter* or just *filter* to denote an operator which convolves and then decimates. A filter operator is defined by the sequence which is convolved with the input. If the filter sequence is finitely supported, we have a *finite impulse response* or *FIR* filter; otherwise we have an *IIR* or *infinite impulse response* filter. We can also project such actions onto periodic sequences, and define *periodized* filters. Filtering is the fundamental arithmetic operations in the discrete wavelet transform.

An individual quadrature filter is not generally invertible; it loses information during the decimation step. However, it is possible to construct a pair complementary filters with each preserving the information lost by the other; the pair can be combined into an invertible operator. Each member of the pair has an *adjoint* operator: when we use filters in pairs to decompose functions and sequences into pieces, it is the adjoint operators which put these pieces back together. The operation is reversible and restores the original signal if we have so-called *exact reconstruction* filters. The pieces will be orthogonal if we have *orthogonal filters* for which the decomposition gives a pair of orthogonal projections which we will define below. Such pairs must satisfy certain algebraic conditions which are completely derived in [3], pp.156–166.

One way to guarantee exact reconstruction is to have "mirror symmetry" of the Fourier transform of each filter about $\xi = \frac{1}{2}$; this leads to what Esteban and Galand [5] first called *quadrature mirror filters* or *QMFs*. Unfortunately, there are no orthogonal exact reconstruction FIR QMFs.

Mintzer [13], Smith and Barnwell [14], and Vetterli [15] found a different symmetry assumption which does allow orthogonal exact reconstruction FIR filters. Smith and Barnwell called these *conjugate quadrature filters* or *CQFs*.

By relaxing the orthogonality condition, Cohen, Daubechies, and Feauveau [2] obtained a large family of *biorthogonal* exact reconstruction filters. Such filters come in two pairs: the analyzing filters which split the signal into two pieces, and the synthesizing filters whose adjoints reassemble it. All of these can be FIRs, and the extra degrees of freedom are very useful to the filter designer.

2.4.1 Filter Action on Sequences

A convolution-decimation operator has at least three incarnations, depending upon the domain of the functions upon which it is defined. We have three different formulas for functions of one real variable, for doubly infinite sequences, and for $2q$-periodic sequences. We will use the term quadrature filter or *QF* to refer to all three, since the domain will usually be obvious from the context.

Suppose that $f = \{f(n) : n \in \mathbf{Z}\}$ is an absolutely summable sequence. We define a *convolution-decimation* operator F and its *adjoint* F^* to be operators acting on doubly infinite sequences, given respectively by the following formulas:

$$Fu(i) = \sum_{j=-\infty}^{\infty} f(2i-j)u(j) = \sum_{j=-\infty}^{\infty} f(j)u(2i-j), \qquad i \in \mathbf{Z}; \tag{15}$$

$$F^*u(j) = \sum_{i=-\infty}^{\infty} \bar{f}(2i-j)u(i) = \begin{cases} \displaystyle\sum_{i=-\infty}^{\infty} \bar{f}(2i)u(i+\frac{j}{2}), & j \in \mathbf{Z} \text{ even}, \\ \displaystyle\sum_{i=-\infty}^{\infty} \bar{f}(2i+1)u(i+\frac{j+1}{2}), & j \in \mathbf{Z} \text{ odd}. \end{cases} \tag{16}$$

If f_{2q} is a $2q$-periodic sequence (*i.e.*, with even period), then it can be used to define a *periodic convolution-decimation* F_{2q} from $2q$-periodic to q-periodic sequences and its *periodic adjoint* F^*_{2q} from q-periodic to $2q$-periodic sequences. These are, respectively, the operators

$$F_{2q}u(i) = \sum_{j=0}^{2q-1} f_{2q}(2i-j)u(j) = \sum_{j=0}^{2q-1} f_{2q}(j)u(2i-j), \qquad 0 \le i < q; \tag{17}$$

and

$$F^*_{2q}u(j) = \sum_{i=0}^{q-1} \bar{f}_{2q}(2i-j)u(i) = \begin{cases} \sum_{i=0}^{q-1} \bar{f}_{2q}(2i)u(i+\frac{j}{2}), & \text{if } j \in [0, 2q-2] \text{ is even,} \\ \sum_{i=0}^{q-1} \bar{f}_{2q}(2i+1)u(i+\frac{j+1}{2}), & \text{if } j \in [1, 2q-1] \text{ is odd.} \end{cases} \tag{18}$$

Periodization commutes with convolution-decimation: we get the same periodic sequence whether we first convolve and decimate an infinite sequence and then periodize the result, or first periodize both the sequence and the filter and then perform a periodic convolution-decimation. The following proposition makes this precise:

Proposition 2.7 $\quad (Fu)_q = F_{2q}u_{2q}$ *and* $(F^*u)_{2q} = F^*_{2q}u_q$.

Proof: This straightforward calculation may be found in [16] on pp.155–156. □

2.4.2 Biorthogonal QFs

A quadruplet H, H', G, G' of convolution-decimation operators or filters is said to form a set of *biorthogonal quadrature filters* or *BQFs* if the filters satisfy the following conditions:

Duality: $H'H^* = G'G^* = I = HH'^* = GG'^*$;

Independence: $G'H^* = H'G^* = 0 = GH'^* = HG'^*$;

Exact reconstruction: $H^*H' + G^*G' = I = H'^*H + G'^*G$;

Normalization: $H\mathbf{1} = H'\mathbf{1} = \sqrt{2}\,\mathbf{1}$ and $G\mathbf{1} = G'\mathbf{1} = \mathbf{0}$, where $\mathbf{1} = \{\ldots, 1, 1, 1, \ldots\}$ is all ones and $\mathbf{0} = \{\ldots, 0, 0, 0, \ldots\}$ is all zeroes.

The first two conditions may be expressed in terms of the filter sequences h, h', g, g' which respectively define H, H', G, G':

$$\sum_k h'(k)\bar{h}(k+2n) = \sum_k g'(k)\bar{g}(k+2n) = \delta(n); \qquad \sum_k g'(k)\bar{h}(k+2n) = \sum_k h'(k)\bar{g}(k+2n) = 0. \tag{19}$$

The normalization condition allows us to say that H and H' are the *low-pass* filters while G and G' are the *high-pass* filters. It may be restated as

$$\sum_k h(k) = \sum_k h'(k) = \sqrt{2}; \qquad \sum_k g(2k) = -\sum_k g(2k+1); \qquad \sum_k g'(2k) = -\sum_k g'(2k+1). \tag{20}$$

Having four operators provides plenty of freedom to construct filters with special properties, but there is also a regular method for constructing the G, G' filters from H, H'. If we have two sequences $\{h(k)\}$ and $\{h'(k)\}$ which satisfy Equation 19, then we can obtain two *conjugate quadrature filter sequences* $\{g(k)\}$ and $\{g'(k)\}$ via the formulas below, using any integer M:

$$g(k) = (-1)^k \bar{h}'(2M+1-k); \qquad g'(k) = (-1)^k \bar{h}(2M+1-k). \tag{21}$$

We also have the following result, which is related to Lemma 12 in [6] and a similar result in [11]:

Lemma 2.8 *The biorthogonal QF conditions imply* $H^*\mathbf{1} = H'^*\mathbf{1} = \frac{1}{\sqrt{2}}\mathbf{1}$.

Proof: With exact reconstruction, $\mathbf{1} = \left(H'^*H + G'^*G\right)\mathbf{1} = \sqrt{2}\,H'^*\mathbf{1}$, since $H\mathbf{1} = \sqrt{2}\,\mathbf{1}$ and $G\mathbf{1} = \mathbf{0}$. Likewise, $\mathbf{1} = (H^*H' + G^*G')\,\mathbf{1} = \sqrt{2}\,H^*\mathbf{1}$, since $H'\mathbf{1} = \sqrt{2}\,\mathbf{1}$ and $G'\mathbf{1} = \mathbf{0}$. □

Remark. The conclusion of Lemma 2.8 may be rewritten as follows:

$$\sum_k h(2k) = \sum_k h(2k+1) = \frac{1}{\sqrt{2}} = \sum_k h'(2k) = \sum_k h'(2k+1). \tag{22}$$

If we have the duality, independence, and exact reconstruction conditions, together with $H\mathbf{1} = H'\mathbf{1} = \sqrt{2}\,\mathbf{1}$ but no normalization on G or G', then at least one of the following must be true:

$$G'\mathbf{1} = \mathbf{0} \text{ and } H^*\mathbf{1} = \frac{1}{\sqrt{2}}\mathbf{1}, \quad \text{or} \quad G\mathbf{1} = \mathbf{0} \text{ and } H'^*\mathbf{1} = \frac{1}{\sqrt{2}}\mathbf{1}.$$

However, the BQF conditions as stated insure that the pairs H, G and H', G' are interchangeable in our analyses.

If H, H', G, G' is a set of biorthogonal QFs, and ρ is any nonzero constant, then $H, H', \bar{\rho}G, \rho^{-1}G'$ is another biorthogonal set. We can use this to normalize the G and G' filters so that

$$\sum_k g(2k) = -\sum_k g(2k+1) = \frac{1}{\sqrt{2}} = \sum_k g'(2k) = -\sum_k g'(2k+1). \tag{23}$$

This will be called the *conventional normalization* for the high-pass filters.

Since $H^*H'H^*H' = H^*H'$ and $G^*G'G^*G' = G^*G'$, the combinations H^*H' and G^*G' are projections although they will not in general be orthogonal projections. That is because they need not be equal to their adjoint projections H'^*H and G'^*G.

An argument similar to the one in Proposition 2.7 shows that periodization of biorthogonal QFs to an even period $2q$ preserves the biorthogonality conditions. Writing h_{2q}, h'_{2q}, g_{2q}, and g'_{2q} for the $2q$-periodizations of h, h', g, and g', respectively, we have

$$\sum_k h'_{2q}(k)\bar{h}_{2q}(k+2n) = \sum_k g'_{2q}(k)\bar{g}_{2q}(k+2n) = \delta(n \bmod q); \quad \sum_k g'_{2q}(k)\bar{h}_{2q}(k+2n) = \sum_k h'_{2q}(k)\bar{g}_{2q}(k+2n) = 0.$$

Here we define the periodized Kronecker delta as follows:

$$\delta(n \bmod q) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} \delta(n+qk) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod q, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Periodization to an even period also preserves the sums over the even and odd indices, and thus Lemma 2.8 remains true if we replace h, h', g, and g' with h_{2q}, h'_{2q}, g_{2q}, and g'_{2q}.

2.4.3 Orthogonal QFs

If $H = H'$ and $G = G'$ in a biorthogonal set of QFs, then the pair H, G is called an *orthogonal quadrature filter* pair. In that case the following conditions hold:

Self-duality: $HH^* = GG^* = I$;

Independence: $GH^* = HG^* = 0$;

Exact reconstruction: $H^*H + G^*G = I$;

Normalization: $H\mathbf{1} = \sqrt{2}\,\mathbf{1}$, where $\mathbf{1} = \{\ldots, 1, 1, 1, \ldots\}$.

We will use the abbreviation *OQF* to refer to one or both elements of such a pair. In this normalization, H is the low-pass filter while G is the high-pass filter.

If H and G are formed respectively from the sequences h and g, the duality and independence conditions satisfied by an OQF pair are equivalent to the following equations:

$$\sum_k h(k)\bar{h}(k+2n) = \sum_k g(k)\bar{g}(k+2n) = \delta(n); \qquad \sum_k g(k)\bar{h}(k+2n) = \sum_k h(k)\bar{g}(k+2n) = 0. \tag{25}$$

For orthogonal QFs, we have a stronger result than Lemma 2.8:

Lemma 2.9 *The orthogonal QF conditions imply that $G\mathbf{1} = 0$, $H^*\mathbf{1} = \frac{1}{\sqrt{2}}\mathbf{1}$ and $|G^*\mathbf{1}| = \frac{1}{\sqrt{2}}\mathbf{1}$.*

Proof: This calculation may be found in [16], pp.159–160. □

If H, G are a pair of orthogonal QFs and ρ is any constant with $|\rho| = 1$, then $H, \rho G$ are also orthogonal QFs. Hence by taking $\rho = \sqrt{2}\sum_k \bar{g}(2k)$ we can arrange that

$$\sum_k g(2k) = -\sum_k g(2k+1) = \frac{1}{\sqrt{2}}. \tag{26}$$

As in Equation 23, this will be called the *conventional normalization* of an orthogonal high-pass filter.

Given h satisfying Equation 25, we can generate a *conjugate* g to satisfy the rest of the orthogonal QF conditions by choosing its coefficients as follows [3], using any integer M:

$$g(n) = (-1)^n \bar{h}(2M + 1 - n), \qquad n \in \mathbf{Z}. \tag{27}$$

Notice that this sequence g is conventionally normalized.

Proposition 2.7 shows that periodization of an orthogonal QF pair to an even period $2q$ preserves the orthogonality conditions, and also preserves the sums over the even and odd indices, and thus Lemma 2.9 remains true if we replace h and g with h_{2q} and g_{2q}.

Self-duality gives $H^*HH^*H = H^*H$ and $G^*GG^*G = G^*G$. Notice that H^*H and G^*G are selfadjoint, so H^*H and G^*G are orthogonal projections.

2.5 Phase Response

We wish to recognize features of the original signal from the coefficients produced by transformations involving QFs, so it is necessary to keep track of which portion of the sequence contributes energy to the filtered sequence.

Suppose that F is a finitely supported filter with filter sequence $f(n)$. For any sequence $u \in \ell^2$, if $Fu(n)$ is large at some index $n \in \mathbf{Z}$, then we can conclude that $u(k)$ is large near the index $k = 2n$. Likewise, if $F^*u(n)$ is large, then there must be significant energy in $u(k)$ near $k = n/2$. We can quantify this assertion of nearness using the support of f, or more generally by computing the position of f and its uncertainty computed with Equations 3 and 5. When the support of f is large, the position method gives a more precise notion of where the analyzed function is concentrated.

Consider what happens when $f(n)$ is concentrated near $n = 2T$:

$$Fu(n) = \sum_{j\in\mathbf{Z}} f(j)u(2n - j) = \sum_{j\in\mathbf{Z}} f(j + 2T))u(2n - j - 2T). \tag{28}$$

Since $f(j + 2T)$ is concentrated about $j = 0$, we can conclude by our previous reasoning that if $Fu(n)$ is large, then $u(k)$ is large when $k \approx 2n - 2T$. Similarly,

$$F^*u(n) = \sum_{j\in\mathbf{Z}} \bar{f}(2j - n)u(j) = \sum_{j\in\mathbf{Z}} \bar{f}(2j - n + 2T))u(j + T). \tag{29}$$

Since $\bar{f}(2j - n + 2T)$ is concentrated about $2j - n = 0$, we conclude that if $F^*u(n)$ is big then $u(j + T)$ must be big where $j \approx n/2$, which implies that $u(k)$ is big when $k \approx \frac{n}{2} + T$.

Decimation by 2 and its adjoint respectively cause the doubling and halving of the indices n to get the locations where u must be large. The translation by T or $-2T$ can be considered a "shift" induced by the filter convolution. We can precisely quantify the location of portions of a signal, measure the shift, and correct for it when interpreting the coefficients produced by applications of F and F^*. We will see that nonsymmetric filters might shift different signals by different amounts, with a variation that can be estimated by a simple expression in the filter coefficients. The details of the shift will be called the *phase response* of the filter.

2.5.1 Shifts for Sequences

The notion of position for a sequence is the same as the one for functions defined in Equation 5, only using sums instead of integrals:

$$c[u] \stackrel{\text{def}}{=} \frac{1}{\|u\|^2} \sum_{k\in\mathbf{Z}} k|u(k)|^2. \tag{30}$$

This quantity, whenever it is finite, may also be called the *center of energy* of the sequence $u \in \ell^2$ to distinguish it from the function case.

The center of energy is the first moment of the probability distribution function (or *pdf*) defined by $|u(n)|^2/\|u\|^2$. We will say that the sequence u is *well-localized* if the second moment of that pdf also exists, namely if

$$\sum_{k\in\mathbf{Z}} k^2|u(k)|^2 = \|ku\|^2 < \infty. \tag{31}$$

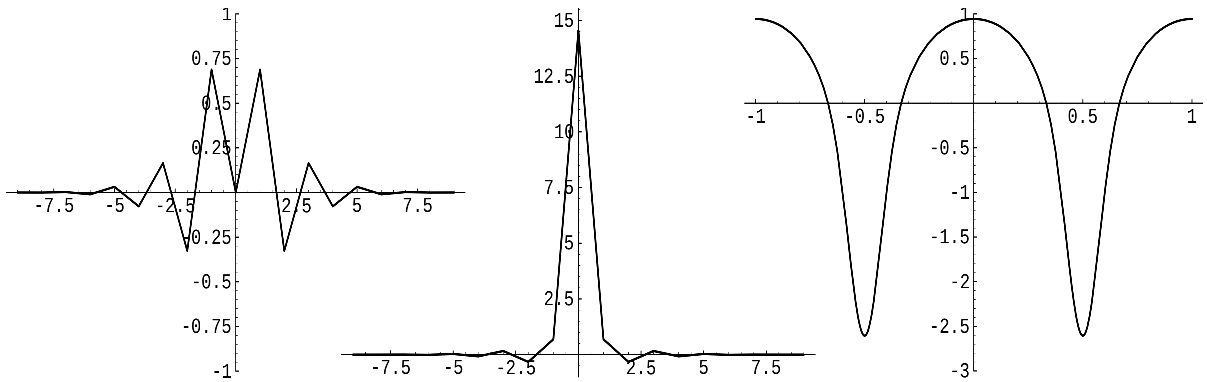

Figure 1: γ^0, γ, and $\hat{\gamma}^0$ for "Beylkin 18" high-pass OQF.

A finite second moment insures that the first moment is also finite, by the Cauchy–Schwarz inequality:

$$\sum_{k\in\mathbf{Z}} k|u(k)|^2 = \langle ku, u\rangle \le \|ku\|\,\|u\| < \infty.$$

If $u \in \ell^2$ is a finitely supported sequence (say in the interval $[a,b]$) then $a \le c[u] \le b$.

Another way of writing $c[u]$ is in Dirac's *bra and ket* notation:

$$\|u\|^2 c[u] = \langle u|X|u\rangle \stackrel{\text{def}}{=} \langle u, Xu\rangle = \sum_{i\in\mathbf{Z}} \bar{u}(i)X(i,j)u(j), \tag{32}$$

where

$$X(i,j) \stackrel{\text{def}}{=} i\delta(i-j) = \operatorname{diag}[\ldots,-2,-1,0,1,2,3,\ldots] = \begin{cases} i, & \text{if } i=j,\\ 0, & \text{if } i\ne j.\end{cases} \tag{33}$$

To simplify the formulas, we will always suppose that $\|u\| = 1$. We can also suppose that f is an orthogonal QF, so $\sum_k \bar{f}(k)f(k+2j) = \delta(j)$. Then $FF^* = I$, F^* is an isometry and F^*F is an orthogonal projection. Since $\|F^*u\| = \|u\| = 1$, we can compute the center of energy of F^*u as $c[F^*u] = \langle F^*u|X|F^*u\rangle = \langle u|FXF^*|u\rangle$. We will call the the double sequence FXF^* between the bra and the ket the *phase response* of the adjoint convolution-decimation operator F^* defined by the filter sequence f. Namely,

$$FXF^*(i,j) = \sum_k kf(2i-k)\bar{f}(2j-k). \tag{34}$$

Now

$$FXF^*(i,j) = \sum_k([i{+}j]+k)f([i{-}j]-k)\bar{f}([j{-}i]-k) \stackrel{\text{def}}{=} 2X(i,j) - C_f(i,j).$$

Here $2X(i,j) = (i+j)\sum_k f([i{-}j]-k)\bar{f}([j{-}i]-k) = 2i\delta(i-j)$ as above, since f is an orthogonal QF, while

$$C_f(i,j) \stackrel{\text{def}}{=} \sum_k kf(k-[i{-}j])\bar{f}(k-[j{-}i]). \tag{35}$$

Thus $c[F^*u] = 2c[u] - \langle u|C_f|u\rangle$. C_f is evidently a convolution matrix: $C_f(i,j) = \gamma(i-j)$ so that $C_f u = \gamma * u$. The function γ is defined by the following formula:

$$\gamma(n) \stackrel{\text{def}}{=} \sum_k kf(k-n)\bar{f}(k+n). \tag{36}$$

From this formula it is easy to see that $\gamma(n) = \bar{\gamma}(-n)$, thus $\hat{\gamma}(\xi) = \hat{\bar{\gamma}}(-\xi) = \overline{\hat{\gamma}}(\xi) \Rightarrow \hat{\gamma} \in \mathbf{R}$. This symmetry of γ makes the matrix C_f selfadjoint. Along its main diagonal, $C_f(i,i) = \gamma(0) = c[f]$. Other diagonals of C_f are constant, and if f is supported in the finite interval $[a,b]$, then $C_f(i,j) = \gamma(i-j) = 0$ for $|i-j| > |b-a|$.

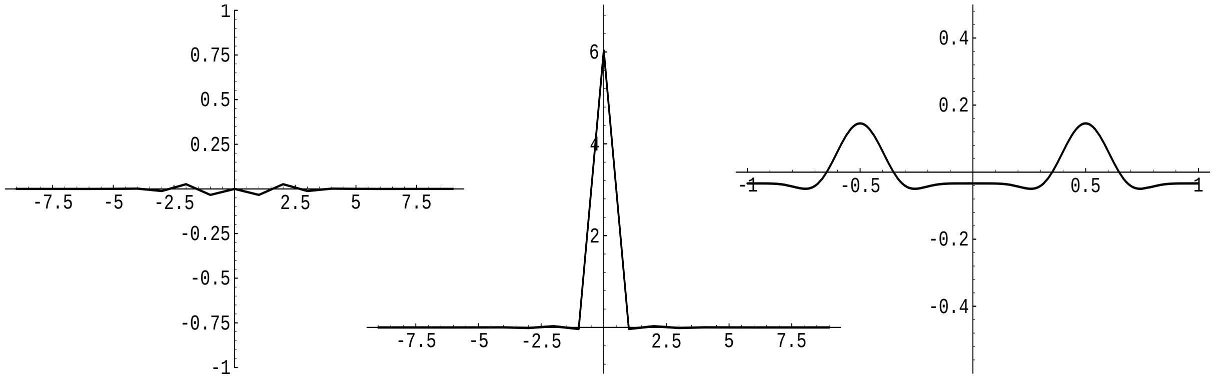

Figure 2: γ^0, γ, and $\hat{\gamma}^0$ for "Coiflet 18" low-pass OQF.

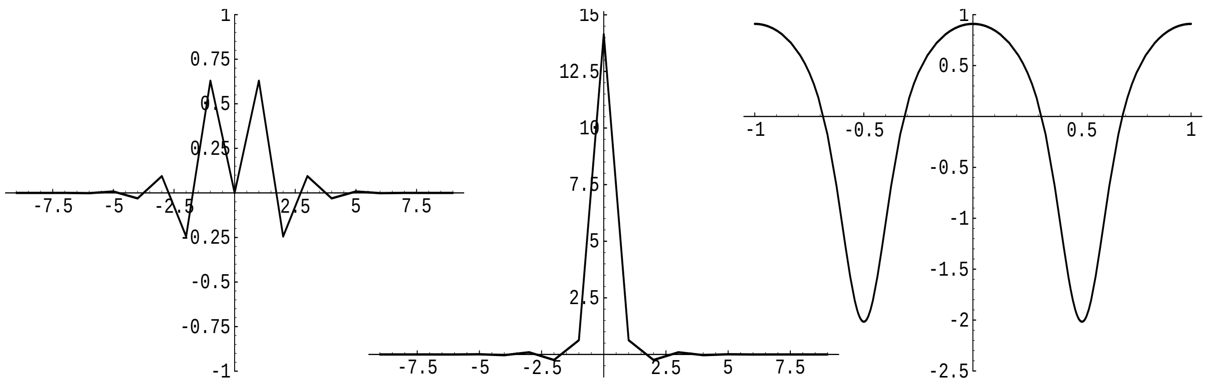

Figure 3: γ^0, γ, and $\hat{\gamma}^0$ for "Daubechies 18" high-pass OQF.

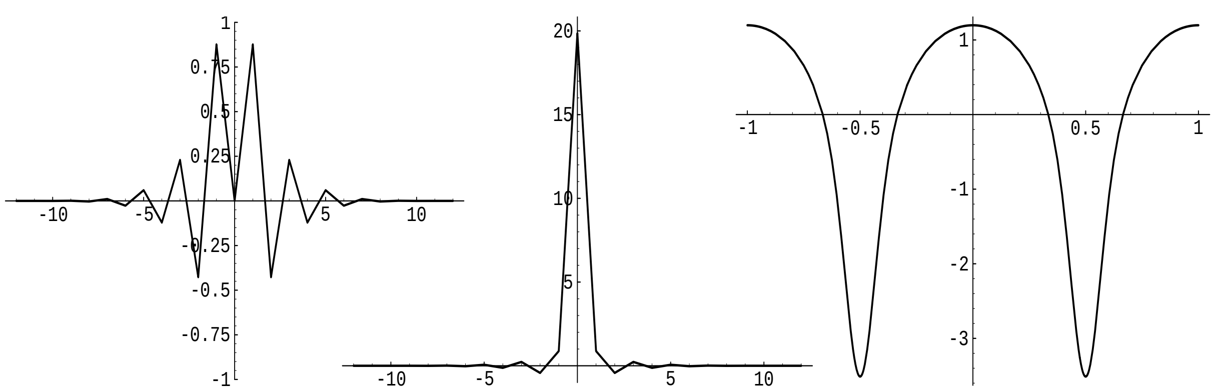

Figure 4: γ^0, γ, and $\hat{\gamma}^0$ for "Vaidyanathan 24" low-pass OQF.

We can subtract the diagonal from C_f by writing $C_f = C_f^0 + c[f]I$, which is the same as the decomposition $\gamma(n) = \gamma^0(n) + c[f]\delta(n)$. This gives a decomposition of the phase response matrix:

$$FXF^* = 2X - c[f]I - C_f^0.$$

Thus FXF^* is multiplication by the linear function $2x - c[f]$ minus convolution with γ^0. We will say that f has a *linear phase response* if $\gamma^0 \equiv 0$.

Proposition 2.10 *Suppose that $f = \{f(n) : n \in \mathbf{Z}\}$ satisfies $\sum_k \bar{f}(k-n)f(k+n) = \delta(n)$ for $n \in \mathbf{Z}$. If f is Hermitean symmetric or antisymmetric about some integer or half integer T, then the phase response of f is linear.*

Proof: We have $f(n) = \pm\bar{f}(2T-n)$ for all $n \in \mathbf{Z}$, taking $+$ in the symmetric case and $-$ in the antisymmetric case. Now $\gamma^0(0) = 0$ for all filters. For $n \neq 0$ we have

$$\begin{aligned}\gamma^0(n) &= \sum_k kf(k-n)\bar{f}(k+n) \;=\; \sum_k k\bar{f}(2T-k+n)f(2T-k-n) \\ &= 2T\sum_k \bar{f}(k+n)f(k-n) - \sum_k k\bar{f}(k+n)f(k-n) \;=\; 0 - \gamma^0(n).\end{aligned}$$

Thus we have $\gamma^0(n) = 0$ for all $n \in \mathbf{Z}$. □

The linear function shifts the center of energy x to $2x - c[f]$, and the convolution operator γ^0 perturbs this by a "deviation" $\langle u, \gamma^0 * u\rangle/\|u\|^2$. We can denote the maximum value of this perturbation by $d[f]$. By Plancherel's theorem and the convolution theorem, the deviation is $\langle \hat{u}, \hat{\gamma}^0\hat{u}\rangle/\|u\|^2$ and its maximum value is given (using Proposition 2.6) by the maximum absolute value of $\hat{\gamma}^0(\xi)$:

$$d[f] = \sup\{|\hat{\gamma}^0(\xi)| : \xi \in [0,1]\}. \tag{37}$$

Now $\gamma^0(n) = \bar{\gamma}^0(-n)$ is symmetric just like γ, so its Fourier transform $\hat{\gamma}^0$ is purely real and can be computed using only cosines as follows:

$$\hat{\gamma}^0(\xi) = 2\sum_{n=1}^{\infty} \gamma(n)\,\cos 2\pi n\xi. \tag{38}$$

The critical points of $\hat{\gamma}^0$ are found by differentiating Equation 38:

$$\hat{\gamma}_0'(\xi) = -4\pi\sum_{n=1}^{\infty} n\gamma(n)\,\sin 2\pi n\xi. \tag{39}$$

It is evident that $\xi = 0$ and $\xi = \frac{1}{2}$ are critical points. For the QFs listed in the appendix, we can show that $|\hat{\gamma}^0(\xi)|$ achieves its maximum at $\xi = \frac{1}{2}$, where

$$\hat{\gamma}^0\left(\frac{1}{2}\right) = 2\sum_{n=1}^{\infty}(-1)^n\gamma(n) = 2\sum_{k=-\infty}^{\infty}\sum_{n=1}^{\infty}(-1)^n kf(k-n)\bar{f}(k+n). \tag{40}$$

Graphs of $\hat{\gamma}^0$ for some examples of orthogonal OQFs can be seen in Figures 1 through 4.

Values of the quantities $c[f]$ and $d[f]$ for example OQFs are listed in Table 1. Notice that if $g(n) = (-1)^n\bar{h}(2M+1-n)$, so that h and g are a conjugate pair of filters, and $|\operatorname{supp} g| = |\operatorname{supp} h| = 2M$ is the length of the filters, then $d[g] = d[h]$ and $c[g]+c[h] = 2M-1$. This also implies that $C_h(i,j) = -C_g(i,j)$, so that the function $\hat{\gamma}^0$ corresponding to the filter h is just the negative of the one corresponding to g.

We can put the preceding formulas together into a single theorem:

Theorem 2.11 (OQF Phase Shifts) *Suppose that $u \in \ell^2$ and that $F : \ell^2 \to \ell^2$ is convolution and decimation by two with an orthogonal QF $f \in \ell^1$. Suppose that $c[u]$ and $c[f]$ both exist. Then*

$$c[F^*u] = 2c[u] - c[f] - \langle u, \gamma^0 * u\rangle/\|u\|^2,$$

where $\gamma^0 \in \ell^2$ is the sequence

$$\gamma^0(n) = \begin{cases} 0, & \text{if } n = 0, \\ \sum_k kf(k-n)\bar{f}(k+n), & \text{if } n \neq 0. \end{cases}$$

The last term satisfies the sharp inequality

$$|\langle u, \gamma^0 * u\rangle| \leq d[f]\,\|u\|^2,$$

f	$\lvert\operatorname{supp} f\rvert$	H or G	$c[f]$	$d[f]$
B	18	H	2.4439712920	2.6048841893
		G	14.5560287079	2.6048841893
C	6	H	3.6160691415	0.4990076823
		G	1.3839308584	0.4990076823
	12	H	4.0342243997	0.0868935216
		G	6.9657756002	0.0868935217
	18	H	6.0336041704	0.1453284669
		G	10.9663958295	0.1453284670
	24	H	8.0333521640	0.1953517707
		G	14.9666478359	0.1953517692
	30	H	10.0333426139	0.2400335062
		G	18.9666573864	0.2400330874
D	2	H	0.5000000000	0.0000000000
		G	0.5000000000	0.0000000000
	4	H	0.8504809471	0.2165063509
		G	2.1495190528	0.2165063509
	6	H	1.1641377716	0.4604317871
		G	3.8358622283	0.4604317871
	8	H	1.4613339067	0.7136488576
		G	5.5386660932	0.7136488576
	10	H	1.7491114972	0.9711171403
		G	7.2508885027	0.9711171403
	12	H	2.0307505738	1.2308332718
		G	8.9692494261	1.2308332718
	14	H	2.3080529576	1.4918354676
		G	10.6919470423	1.4918354676
	16	H	2.5821186257	1.7536045071
		G	12.4178813742	1.7536045071
	18	H	2.8536703515	2.0158368941
		G	14.1463296483	2.0158368941
	20	H	3.1232095535	2.2783448731
		G	15.8767904464	2.2783448731
V	24	H	19.8624838621	3.5116226595
		G	3.1375161379	3.5116226595

Table 1: Center-of-energy shifts and errors for some example OQFs.

where

$$d[f] = 2\left|\sum_{k=-\infty}^{\infty}\sum_{n=1}^{\infty}(-1)^n k f(k-n)\bar{f}(k+n)\right|.$$

□

If $d[f]$ is small, then we can safely ignore the deviation of F^*u from a pure shift of u by $c[f]$. In that case, we will say that $c[F^*u] \approx 2c[u] - c[f]$ and $c[Fu] \approx \frac{1}{2}c[u] + \frac{1}{2}c[f]$. We note that the "C" filters have the smallest errors $d[f]$; these are the filters to use if we wish to extract reasonably accurate position information.

If we apply a succession of filters $F_1^*F_2^*\cdots F_L^*$, then by induction on L we can compute the shifts as follows:

$$c[F_1^*F_2^*\cdots F_L^*u] = 2^L c[u] - 2^{L-1}c[f_L] - \cdots - 2^1 c[f_2] - c[f_1] - \epsilon^*, \tag{41}$$

where

$$|\epsilon^*| \le 2^{L-1}d[f_L] + \cdots + 2^1 d[f_2] + d[f_1]. \tag{42}$$

Similarly, if $v = F_1^*F_2^*\cdots F_L^*u$, so that $F_L\cdots F_2F_1v = u$, then the following holds:

$$c[F_L\cdots F_2F_1v] = 2^{-L}c[v] + 2^{-L}c[f_1] + 2^{-L+1}c[f_2] + \cdots + 2^{-1}c[f_L] + \epsilon, \tag{43}$$

where

$$|\epsilon| \le 2^{-1}d[f_L] + \cdots + 2^{-L+1}d[f_2] + 2^{-L}d[f_1]. \tag{44}$$

Now suppose that (h, g) is a conjugate pair of OQFs, so that $f_i \in \{h, g\}$ for each $i = 1, 2, \ldots, L$. Then $d[f_i]$ is constantly $d[h]$ and we have the simpler estimates for the deviation from a pure shift:

$$|\epsilon^*| \le (2^L - 1)\,d[h] \approx 2^L d[h] \qquad \text{and} \qquad |\epsilon| \le (1 - 2^{-L})\,d[h] \approx d[h]. \tag{45}$$

Suppose that we encode the sequence of filters $F_1^*F_2^*\cdots F_L^*$ as the integer $b = b_1 2^{L-1} + b_2 2^{L-2} + \cdots + b_L 2^0$, where

$$b_k = \begin{cases} 0, & \text{if } F_k = H; \\ 1, & \text{if } F_k = G. \end{cases} \tag{46}$$

Then we can write $c[f_k] = b_k c[g] + (1 - b_k)c[h] = c[h] + b_k(c[g] - c[h])$. Notice that the bit-reversal of b, considered as an s-bit binary integer, is the integer $b' = b_1 2^0 + b_2 2^1 + \cdots + b_L 2^{L-1}$. This simplifies the formula for the phase shift as follows:

Corollary 2.12 *If h and g are a conjugate pair of OQFs with centers of energy $c[h]$ and $c[g]$, respectively, then*

$$c[F_1^*F_2^*\cdots F_L^*u] = 2^L c[u] - \left(2^L - 1\right)c[h] - (c[g] - c[h])\,b' - \epsilon^*, \tag{47}$$

where $|\epsilon^*| \le (2^L - 1)\,d[h]$ and $b = b_1 2^{L-1} + b_2 2^{L-2} + \cdots + b_L$ encodes the sequence of filters as in Equation 46, and b' is the bit-reversal of b considered as an L-bit binary integer.

Proof: We observe that

$$\begin{aligned} c[F_1^*F_2^*\cdots F_L^*u] &= 2^L c[u] - \sum_{k=1}^{L} 2^{L-k}\left[c[h] + b_{L-k+1}(c[g] - c[h])\right] - \epsilon^* \\ &= 2^L c[u] - c[h]\sum_{s=0}^{L-1} 2^s - (c[g] - c[h])\sum_{s=0}^{L-1} b_{s+1}2^s - \epsilon^* \\ &= 2^L c[u] - \left(2^L - 1\right)c[h] - (c[g] - c[h])b' - \epsilon^*. \end{aligned}$$

The estimate on ϵ^* follows from Equation 45. □

2.5.2 Shifts in the Periodic Case

Defining a center of energy for a periodic signal is problematic. However, if a periodic signal contains a component with a distinguishable scale much shorter than the period, then it may be desirable to locate this component within the period. If the component is characterized by a large amplitude found by filtering, then we can locate it by interpreting the "position" information of the filter output. We must adjust this position information by the center-of-energy shift caused by filtering, and allow for the deviation due to phase nonlinearity. In the periodic case, the shift can be approximated by a cyclic permutation of the output coefficients.

We can compute the center of energy of a nonzero q-periodic sequence u_q as follows:

$$c[u_q] = \frac{1}{\|u_q\|^2} \sum_{k=0}^{q-1} k|u_q(k)|^2.$$

Since $c[u_q]$ is a convex combination of $0, 1, \ldots, q-1$, we have $0 \leq c[u_q] \leq q-1$. Now suppose that u_q is the q-periodization of u and that all but ϵ of the energy in the sequence u comes from coefficients in one period interval $J_0 \stackrel{\text{def}}{=} [j_0 q, j_0 q + q - 1]$, for some integer j_0 and some positive $\epsilon \ll 1$. We must also suppose that u has a finite position uncertainty which is less than q. These conditions may be succinctly combined into the following:

$$\left(\sum_{j \notin J_0} \left[j - (j_0 + \frac{1}{2})q \right]^2 |u(j)|^2 \right)^{\frac{1}{2}} < q\epsilon \|u\|. \tag{48}$$

Equation 48 and some straightforward computations (see [16], pp.172–174) produce the following inequalities:

$$\left| \|u_q\|^2 \left[c[u_q] - \frac{q}{2} \right] - \|u\|^2 \left[c[u] - j_0 q - \frac{q}{2} \right] \right| < 2q\epsilon \left(1 + 5\epsilon\right) \|u\|^2; \qquad \left| \|u_q\|^2 - \|u\|^2 \right| < 4\epsilon \left(1 + 5\epsilon\right) \|u\|^2. \tag{49}$$

We can replace $\|u_q\|^2$ with $\|u\|^2$ in the left inequality of 49:

$$|c[u_q] - c[u] + j_0 q| < 4q\epsilon \left(1 + 5\epsilon\right). \tag{50}$$

Hence, if almost all of the energy of u is concentrated on an interval of length q, then transient features of u have a scale smaller than q and will become transient features of u_q upon q-periodization. These will be located at nearly the same position modulo q as features of u, and we can use the following approximation to locate the center of energy of a periodized sequence to within one index:

$$c[u_q] \stackrel{\text{def}}{=} c[u] \bmod q. \tag{51}$$

We interpret the expression "$x \bmod q$" to mean the unique real number x' in the interval $[0, q[$ such that $x = x' + nq$ for some integer n.

We can use Proposition 2.7 to compute the following approximation:

$$c[F^*_{2q} u_q] = c[(F^* u)_{2q}] = c[F^* u] \bmod 2q = 2c[u] - c[f] - \langle u, \gamma^0 * u \rangle / \|u\|^2 \bmod 2q.$$

Now $\langle u, \gamma^0 * u \rangle / \|u\|^2$ is bounded by $d[f]$ so we plan to ignore it as before, though we must still verify that the OQFs satisfy Equation 48 with sufficiently small ϵ. Table 2 shows the value of ϵ for a few example OQFs and a few example periodizations. In all cases $\epsilon < 1$, so the table lists only the digits after the decimal point.

Since there is no unique way to *deperiodize* u_q to an infinite sequence u, it is necessary to adopt a convention. The simplest would be the following:

$$u(n) = \begin{cases} u_q(n), & \text{if } 0 \leq n < q, \\ 0, & \text{otherwise.} \end{cases} \tag{52}$$

3 Wavelet Registration

We now consider the second problem: an algorithm for finding the *best shift* for a periodic discrete wavelet transform. Our procedure is to find which periodic shift of a signal produces the lowest *information cost*.

3.1 Information Cost

Before we can define an optimum representation we need to have a notion of *information cost*, or the expense of storing the chosen representation. So, define an *information cost functional* on sequences of real (or complex) numbers to be any real-valued functional M satisfying the additivity condition below:

$$M(u) = \sum_{k \in \mathbf{Z}} \mu(|u(k)|); \qquad \mu(0) = 0. \tag{53}$$

Here μ is a real-valued function defined on $[0, \infty)$. We suppose that $\sum_k \mu(|u(k)|)$ converges absolutely; then M will be invariant under rearrangements of the sequence u. Also, M is not changed if we replace $u(k)$ by $-u(k)$ for some

f	$\lvert\operatorname{supp} f\rvert$	H or G	$q=2$ $q=16$	$q=4$ $q=18$	$q=6$ $q=20$	$q=8$ $q=22$	$q=10$ $q=24$	$q=12$ $q=26$	$q=14$ $q=28$
B	18	H	.703612 .001415	.279300	.142238	.074249	.033688	.014072	.005406
		G	.734120 .001590	.324821	.163452	.087139	.038976	.016137	.006156
C	6	H	.247013	.102745					
		G	.268885	.069768					
	12	H	.263115	.072831	.033281	.010694	.001009		
		G	.251051	.070544	.028711	.009039	.001205		
	18	H	.299435 .000040	.100032	.052849	.018963	.007231	.002661	.000621
		G	.291211 .000045	.098243	.046702	.017889	.007332	.002556	.000708
	24	H	.329096 .000890	.120402 .000331	.065564 .000036	.027330 .000002	.014121	.005809	.002328
		G	.322880 .000936	.119051 .000367	.060292 .000039	.027004 .000002	.013983	.005754	.002531
	30	H	.354113 .002035	.136558 .000958	.075916 .000291	.035107 .000138	.020482 .000026	.009303 .000002	.004743 .000000
		G	.349093 .002111	.135636 .001051	.071338 .000285	.035330 .000134	.020121 .000024	.009401 .000002	.005009 .000000
D	4	H	.171193						
		G	.273971						
	6	H	.304120	.050230					
		G	.259392	.073125					
	8	H	.308900	.102651	.017895				
		G	.323009	.122720	.023634				
	10	H	.342554	.135552	.040530	.006627			
		G	.449328	.116023	.053618	.008251			
	12	H	.422494	.137647	.058646	.016224	.002475		
		G	.463486	.160047	.064599	.020210	.002964		
	14	H	.524235	.169394	.072909	.023686	.006412	.000924	
		G	.508880	.223013	.076843	.029062	.007680	.001077	
	16	H	.524480	.210433	.085366	.032061	.009408	.002489	.000344
		G	.587024	.220427	.103528	.038321	.011119	.002899	.000393
	18	H	.564454 .000128	.243878	.102607	.045068	.014338	.003662	.000948
		G	.636888 .000144	.238832	.128066	.050826	.016666	.004213	.001082
	20	H	.634131 .000354	.248979 .000047	.120135	.051443	.024453	.006775	.001411
		G	.672192 .000398	.282813 .000053	.138670	.060597	.025714	.007739	.001591
V	24	H	.872011 .006270	.390176 .001937	.217686 .000629	.116186 .000191	.062451	.036782	.017151
		G	.829783 .005653	.355441 .001764	.190529 .000574	.101064 .000175	.057180	.034695	.015266

Table 2: Concentration of energy for some example orthogonal QFs.

k, or, in the case of complex-valued sequences u, if we multiply the elements of the sequence by complex constants of modulus 1. We take M to be real-valued so that we can compare two sequences u and v by comparing $M(u)$ and $M(v)$.

For each $x \in X$ we can take $u(k) = B^*x(k) = \langle b_k, x\rangle$, where $b_k \in B$ is the k^{th} vector in the basis $B \in \mathcal{B}$. In the finite-rank case, we can think of b_k as the k^{th} column of the matrix B, which is taken with respect to a standard basis of X. The information cost of representing x in the basis B is then $M(B^*x)$. This defines a functional $\mathcal{M}_x$ on the set of bases $\mathcal{B}$ for X:

$$\mathcal{M}_x : \mathcal{B} \to \mathbf{R}; \qquad B \mapsto M(B^*x). \tag{54}$$

This will be called the *M-information cost of x in the basis B*.

We define the *best basis* for $x \in X$, relative to a collection $\mathcal{B}$ of bases for X and an information cost functional M, to be that $B \in \mathcal{B}$ for which $M(B^*x)$ is minimal. If we take $\mathcal{B}$ to be the complete set of orthonormal bases for X, then $\mathcal{M}_x$ defines a functional on the group $\mathbf{O}(X)$ of orthogonal (or unitary) linear transformations of X. We can use the group structure to construct *information cost metrics* and interpret our algorithms geometrically.

We can define all sorts of real-valued functionals M, but the most useful are those that measure concentration. By this we mean that M should be large when elements of the sequence are roughly the same size and small when all but a few elements are negligible. This property should hold on the unit sphere in ℓ^2 if we are comparing orthonormal bases, or on a spherical shell in ℓ^2 if we are comparing Riesz bases or frames.

Some examples of information cost functionals are:

- *Number above a threshold*

  We can set an arbitrary threshold ϵ and count the elements in the sequence x whose absolute value exceeds ϵ. *I.e.*, set

  $$\mu(w) = \begin{cases} |w|, & \text{if } |w| \geq \epsilon, \\ 0, & \text{if } |w| < \epsilon. \end{cases}$$

  This information cost functional counts the number of sequence elements needed to transmit the signal to a receiver with precision threshold ϵ.

- *Concentration in ℓ^p*

  Choose an arbitrary $0 < p < 2$ and set $\mu(w) = |w|^p$ so that $M(u) = \|\{u\}\|_p^p$. Note that if we have two sequences of equal energy $\|u\| = \|v\|$ but $M(u) < M(v)$, then u has more of its energy concentrated into fewer elements.

- *Entropy*

  Define the *entropy* of a vector $u = \{u(k)\}$ by

  $$\mathcal{H}(u) = \sum_k p(k) \log \frac{1}{p(k)}, \tag{55}$$

  where $p(k) = |u(k)|^2/\|u\|^2$ is the normalized energy of the k^{th} element of the sequence, and we set $p \log \frac{1}{p} = 0$ if $p = 0$. This is the entropy of the probability distribution function (or *pdf*) given by p. It is not an information cost functional, but the functional $l(u) = \sum_k |u(k)|^2 \log(1/|u(k)|^2)$ is. By the relation

  $$\mathcal{H}(u) = \|u\|^{-2} l(u) + \log \|u\|^2, \tag{56}$$

  minimizing l over a set of equal length vectors u minimizes $\mathcal{H}$ on that set.

- *Logarithm of energy*

  Let $M(u) = \sum_{k=1}^N \log |u(k)|^2$. This may be interpreted as the entropy of a Gauss-Markov process $k \mapsto u(k)$ which produces N-vectors whose coordinates have variances $\sigma_1^2 = |u(1)|^2, \ldots, \sigma_N^2 = |u(N)|^2$. We must assume that there are no unchanging components in the process, *i.e.*, that $\sigma_k^2 \neq 0$ for all $k = 1, \ldots, N$. Minimizing $M(u)$ over $B \in \mathbf{O}(X)$ finds the *Karhunen–Loève basis* for the process; minimizing over a "fast" library $\mathcal{B}$ finds the best "fast" approximation to the Karhunen–Loève basis.

3.1.1 Entropy, Information, and Theoretical Dimension

Suppose that $\{x(n)\}_{n=1}^\infty$ belongs to both L^2 and $L^2 \log L$. If $x(n) = 0$ for all sufficiently large n, then in fact the signal is finite-dimensional. Generalizing this notion, we can compare sequences by their rate of decay, *i.e.*, the rate at which their elements become negligible if they are rearranged in decreasing order.

We define the theoretical dimension of a sequence $\{x(n) : n \in \mathbf{Z}\}$ to be

$$d = \exp\left(\sum_n p(n) \log \frac{1}{p(n)} \right) \tag{57}$$

where $p(n) = |x(n)|^2/\|x\|^2$. Note that $d = \exp \mathcal{H}(x)$ where $\mathcal{H}(x)$, defined in Equation 55 above, is the entropy of the sequence x.

3.1.2 Searching for Minimum Cost

Beylkin in [1] observed earlier that computing the periodic discrete wavelet transform of all N circulant shifts of an N-point periodic signal requires computing only $N \log_2 N$ coefficients. If we build a complete binary tree with information cost tags computed from from appropriate subsets of the shifted coefficients, then the best complete branch will give a representation of the circulant shift which yields the lowest cost transform. After solving the technical problem of ties, the computed shift can be used as a *registration point* for the signal.

The first step is to build a binary tree of the information costs of the wavelet subspaces computed with all circulant shifts. We write the cost of a node of the tree into an auxiliary variable attached to the node, which will later be added together with the other nodes along the branch to give a "branch" cost. We also assume that the output array is at least $q/2$ elements long, to accommodate the intermediate outputs of convolution and decimation. The algorithm is implemented recursively as follows:

shiftscosts(output y; input x; parameter q): *Costs of circulant shifts*

- If $q \leq 1$ then return (this is the recursion termination condition).
- Convolve-decimate the q-periodic input sequence $\{x(1), \ldots, x(q)\}$ to a $q/2$-periodic output sequence $\{y(1), \ldots, y(q/2)\}$ using the high-pass filter G.
- Compute the information cost of y and store it.
- Convolve-decimate the q-periodic input sequence $\{x(1), \ldots, x(q)\}$ to a $q/2$-periodic output sequence $\{y(1), \ldots, y(q/2)\}$ using the low-pass filter H.
- Apply **shiftscosts** to the $q/2$-periodic sequence $\{y(1), y(2), \ldots, y(q/2)\}$.
- Apply **shiftscosts** to the $q/2$-periodic sequence $\{y(2), \ldots, y(q/2), y(1)\}$.

The function `shiftscosts` can also be used to accumulate the costs of a a whole branch into its leaf at the same time that we compute the coefficients, as we descend. One of the inputs to the function is q, and we assume that the input sequence is q-periodic and registered at 0. Then the information cost of a 2^L-point discrete periodic wavelet transform shifted by T will be found in the node at level L whose block index is the bit-reverse of T. We can extract these values with a utility function, then use a bubble sort to find the least one while searching in bit-reversed order, and return its index. This finds the least circulant shift which yields the minimal information cost.

To *register* a periodic signal, we compute the registration point and then circularly shift the signal so that the registration point becomes index zero. It is also possible to avoid the use of a binary tree data structure by directly writing the costs of circulant-shifted wavelet coefficients to an array.

Wavelet registration works because the information cost of the wavelet subspace W_k of a 2^L-periodic signal is a 2^k-periodic function for each $0 \leq k \leq L$. Thus the information cost in the node at level k, block n is the information cost of W_k with a circulant shift by $n' \pmod{2^k}$, where n' is the length k bit-reversal of n. A branch to a leaf node at block index n contains the wavelet subspaces $W_1, \ldots, W_L$ of the periodic discrete wavelet transform with shift n'. The scaling subspace V_L in the periodic case always contains the unweighted average of the coefficients, which is invariant under shifts.

We can define a *shift cost function* for a 2^L-periodic signal to be the map $f(n) = c_{n'L}$, the information cost in the tag of the costs tree at level L and block index n', the bit-reverse of n.

Two 2^L-point signals whose principal difference is a circulant shift can be compared by cross-correlating their shift cost functions. This is an alternative to traditional cross-correlation of the signals themselves, or multiscale cross-correlation of their wavelet and scaling subspaces as done in [9].

References

[1] Gregory Beylkin. On the representation of operators in bases of compactly supported wavelets. *SIAM Journal of Numerical Analysis*, 6-6:1716–1740, 1992.

[2] Albert Cohen, Ingrid Daubechies, and Jean-Christophe Feauveau. Biorthogonal bases of compactly supported wavelets. *Communications on Pure and Applied Mathematics*, 45:485–500, 1992.

[3] Ingrid Daubechies. Orthonormal bases of compactly supported wavelets. *Communications on Pure and Applied Mathematics*, XLI:909–996, 1988.

[4] Ingrid Daubechies. *Ten Lectures on Wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. SIAM Press, Philadelphia, 1992.

[5] D. Esteban and C. Galand. Application of quadrature mirror filters to split band voice coding systems. In *Proceedings of IEEE ICASSP-77*, pages 191–195, Washington, D.C., May 1977.

[6] Ramesh A. Gopinath and C. Sidney Burrus. Wavelet transforms and filter banks. In Charles K. Chui, editor, *Wavelets–A Tutorial in Theory and Applications*, pages 603–654. Academic Press, Boston, 1992.

[7] Alexander Grossmann and Jean Morlet. Decomposition of Hardy functions into square-integrable wavelets of constant shape. *SIAM Journal of Mathematical Analysis*, 8:4, 1985.

[8] Philippe Guillemain, Richard Kronland-Martinet, and B. Martens. Estimation of spectral lines with the help of the wavelet transform—applications in NMR spectroscopy. In Yves Meyer, editor, *Wavelets and Applications*, Proceedings of the International Conference "Wavelets and Applications" Marseille, May 1989, pages 38–60. Masson, Paris, 1992.

[9] P. G. Hjorth, Lars F. Villemoes, J. Teuber, and R. Florentin-Nielsen. Wavelet analysis of 'double quasar' flux data. *Astronomy and Astrophysics*, 255:20–23, 1992.

[10] Richard Kronland-Martinet, Jean Morlet, and Alexander Grossmann. Analysis of sound patterns through wavelet transforms. *International Journal of Pattern Recognition and Arificial Intelligence*, 1(2):273–302, 1987.

[11] Wayne M. Lawton. Necessary and sufficient conditions for existence of orthonormal wavelet bases. *Journal of Mathematical Physics*, 32:57–61, 1991.

[12] Stéphane G. Mallat and W. L. Hwang. Singularity detection and processing with wavelets. *IEEE Transactions on Information Theory*, 32:617–643, March 1992.

[13] F. Mintzer. Filters for distortion-free two-band multirate filter banks. *IEEE Transactions on Acoustics, Speech and Signal Processing*, 33:626–630, 1985.

[14] M. J. T. Smith and T. P. Barnwell III. A procedure for designing exact reconstruction filter banks for tree-structured sub-band coders. In *Proceedings of IEEE ICASSP-84*, pages 27.1.1–27.1.4, San Diego, CA, March 1984.

[15] Martin Vetterli. Filter banks allowing perfect reconstruction. *Signal Processing*, 10(3):219–244, 1986.

[16] Mladen Victor Wickerhauser. *Adapted Wavelet Analysis from Theory to Software.* AK Peters, Ltd., Wellesley, Massachusetts, 9 May 1994. 486+xii pages, with diskette. To appear.