

## PUTNAM PROBLEMS

### GEOMETRY

**2008-B-1.** What is the maximum number of rational points that can be on a circle in  $\mathbf{R}^2$  whose centre is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

**2007-A-6.** A *triangulation*  $\mathfrak{T}$  of a polygon  $P$  is a finite collection of triangles whose union is  $P$ , and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side of  $P$  is a side of exactly one triangle in  $\mathfrak{T}$ . Say that  $\mathfrak{T}$  is *admissible* if every internal vertex is shared by 6 or more triangles. [An example is given.] Prove that there is an integer  $M_n$ , depending only on  $n$ , such that any admissible triangulation of a polygon  $P$  with  $n$  sides has at most  $M_n$  triangles.

**2006-B-3.** Let  $S$  be a finite set of points in the plane. A linear partition of  $S$  is an unordered pair  $\{A, B\}$  of subsets of  $S$  such that  $A \cup B = S$ ,  $A \cap B = \emptyset$ , and  $A$  and  $B$  lie on opposite sides of some straight line disjoint from  $S$  ( $A$  or  $B$  may be empty). Let  $L_S$  be the number of linear partitions of  $S$ . For each positive integer  $n$ , find the maximum of  $L_S$  over all sets  $S$  of  $n$  points.

**2004-A-2.** For  $i = 1, 2$ , let  $T_i$  be a triangle with side lengths  $a_i, b_i, c_i$  and area  $A_i$ . Suppose that  $a_1 \leq a_2$ ,  $b_1 \leq b_2$ ,  $c_1 \leq c_2$ , and that  $T_2$  is an acute triangle. Does it follow that  $A_1 \leq A_2$ ?

**2003-B-5.** Let  $A, B$  and  $C$  be equidistant points on the circumference of a circle of unit radius centered at  $O$ , and let  $P$  be any point in the circle's interior. Let  $a, b, c$  be the distances from  $P$  to  $A, B, C$  respectively. Show that there is a triangle with side lengths  $a, b, c$ , and that the area of this triangle depends only on the distance from  $P$  to  $O$ .

**2002-A-2.** Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

**2001-A-4.** Triangle  $ABC$  has area 1. Points  $E, F, G$  lie, respectively, on sides  $BC, CA, AB$  such that  $AE$  bisects  $BF$  at point  $R$ ,  $BF$  bisects  $CG$  at point  $S$ , and  $CG$  bisects  $AE$  at point  $T$ . Find the area of triangle  $RST$ .

**2001-A-6.** Can an arc of a parabola inside a circle of radius 1 have length greater than 4?

**2000-A-3.** The octagon  $P_1P_2P_3P_4P_5P_6P_7P_8$  is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon  $P_1P_3P_5P_7$  is a square of area 5 and that the polygon  $P_2P_4P_6P_8$  is a rectangle of area 4, find the maximum possible area of the octagon.

**2000-A-5.** Three distinct points with integer coefficients lie in the plane on a circle of radius  $r > 0$ . Show that two of these points are separated by a distance of at least  $r^{1/3}$ .

**2000-B-6.** Let  $B$  be a set of more than  $2^{n+1}/n$  distinct points with coordinates of the form  $(\pm 1, \pm 1, \dots, \pm 1)$  in  $n$ -dimensional space, with  $n \geq 3$ . Show that there are three distinct points in  $B$  which are vertices of an equilateral triangle. ■

**1999-B-1.** Right triangle  $ABC$  has right angle at  $C$  and  $\angle BAC = \theta$ ; the point  $D$  is chosen on  $AB$  so that  $|AC| = |AD| = 1$ ; the point  $E$  is chosen on  $BC$  so that  $\angle CDE = \theta$ . The perpendicular to  $BC$  at  $E$  meets  $AB$  at  $F$ . Evaluate  $\lim_{\theta \rightarrow 0} |EF|$ . [Here  $|PQ|$  denotes the length of the segment  $PQ$ .]

**1998-A-1.** A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

**1998-A-2.** Let  $s$  be any arc of the unit circle lying entirely in the first quadrant. Let  $A$  be the area

of the region lying below  $s$  and above the  $x$ -axis and  $B$  be the area of the region lying to the right of the  $y$ -axis and to the left of  $s$ . Prove that  $A + B$  depends only on the arc length, and not on the position of  $s$ .

**1998-A-5.** Let  $\mathfrak{F}$  be a finite collection of open discs in  $\mathbf{R}^2$  whose union contains a set  $E \subseteq \mathbf{R}^2$ . Show that there is a pairwise disjoint subcollection  $D_1, D_2, \dots, D_n$  in  $\mathfrak{F}$  such that

$$\bigcup_{j=1}^n 3D_j \supseteq E .$$

Here, if  $D$  is the disc of radius  $r$  and center  $P$ , then  $3D$  is the disc of radius  $3r$  and center  $P$ .

**1998-A-6.** Let  $A, B, C$  denote distinct points with integer coordinates in  $\mathbf{R}^2$ . Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1 ,$$

then  $A, B, C$  are three vertices of a square. Here  $[XY]$  is the length of segment  $XY$  and  $[ABC]$  is the area of triangle  $ABC$ .

**1998-B-2.** Given a point  $(a, b)$  with  $0 < b < a$ , determine the minimum perimeter of a triangle with one vertex at  $(a, b)$ , one on the  $x$ -axis, and one on the line  $y = x$ . You may assume that a triangle of minimum perimeter exists.

**1998-B-3.** Let  $H$  be the unit hemisphere  $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ ,  $C$  the unit circle  $\{(x, y, 0) : x^2 + y^2 = 1\}$ , and  $P$  a regular pentagon inscribed in  $C$ . Determine the surface area of that portion of  $H$  lying over the planar region inside  $P$ , and write your answer in the form  $A \sin \alpha + B \cos \beta$ , where  $A, B, \alpha$  and  $\beta$  are real numbers.

**1997-A-1.** A rectangle,  $HOMF$ , has sides  $HO = 11$  and  $OM = 5$ . A triangle  $ABC$  has  $H$  as the intersection of the altitudes,  $O$  the center of the circumscribed circle,  $M$  the midpoint of  $BC$ , and  $F$  the foot of the altitude from  $A$ . What is the length of  $BC$ ?

**1997-B-6.** The dissection of the  $3 - 4 - 5$  triangle shown below has diameter  $5/2$ .

Find the least diameter of a dissection of this triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between pairs of points belonging to the same part.)

**1996-A-1.** Find the least number  $A$  such that for any two squares of combined area 1, a rectangle of area  $A$  exists such that the two squares can be packed into that rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangles.

**1996-A-2.** Let  $C_1$  and  $C_2$  be circles whose centers are 10 units apart and whose radii are 1 and 3. Find, with proof, the locus of all points  $M$  for which there exist points  $x$  on  $C_1$  and  $Y$  on  $C_2$  such that  $M$  is the midpoint of the lines segment  $XY$ .

**1996-B-6.** Let  $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$  be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers  $x$  and  $y$  such that

$$(a_1, b_1)x^{a_1}y^{b_1} + (a_2, b_2)x^{a_2}y^{b_2} + \dots + (a_n, b_n)x^{a_n}y^{b_n} = (0, 0) \quad .$$

**1995-B-2.** An ellipse, whose semi-axes have lengths  $a$  and  $b$ , rolls without slipping on the curve  $y = c \sin(x/a)$ . How are  $a, b, c$  related, given that the ellipse completes one revolution when it traverses one period of the curve?