# 5th Putnam Practice

Compton Hall, Room 241, 4-6pm

21 October 2011

PROBABILITY

1985–B–4 Let $C$ be the unit circle $x^2+y^2=1$. A point $p$ is chosen randomly on the circumference $C$ and another point $q$ is chosen randomly from the interior of $C$ (these points are chosen independently and uniformly over their domains). Let $R$ be the rectangle with sides parallel to the $x$ and $y$-axes with diagonal $pq$. What is the probability that no point of $R$ lies outside of $C$?

1989–B–1 A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b}+c}{d}$, where $a$, $b$, $c$, $d$ are integers.

1992–A-6 Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

1993–B–2 Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and $n$ cards are dealt to each of two players. Beginning with $A$, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both $A$ and $B$, what is the probability that $A$ wins?

1993–B–3 Two real numbers $x$ and $y$ are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to $x/y$ is even? Express the answer in the form $r+s\pi$, where $r$ and $s$ are rational numbers.

2001–A–2 You have coins $C_1, C_2, \ldots, C_n$. For each $k$, $C_k$ is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the $n$ coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of $n$.

2002–B–1 Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

2002–B–4 An integer $n$, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select $n$ in an **odd** number of guesses. After each incorrect guess, you are informed whether $n$ is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

2004–A–5 An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, $p$ and $q$, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at $p$ and ending at $q$, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

2005–A–6 Let $n$ be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are $n$ randomly, independently and uniformly, chosen points on a circle. Consider the convex $n$-gon whose vertices are $P_i$. What is the probability that at least one of the vertex angles of this polygon is acute?

2006–A–6 Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

2007–A–3 Let $k$ be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

CALCULUS

1986–A–1 Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers $x$ satisfying $x^4 + 36 \leq 13x^2$.

1988–A–2 A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval $(a, b)$ and a nonzero function $g$ defined on $(a, b)$ such that this wrong product rule is true for $x$ in $(a, b)$.

1985–A–2 Let $T$ be an acute triangle. Inscribe a rectangle $R$ in $T$ with one side along a side of $T$. Then inscribe a rectangle $S$ in the triangle formed by the side of $R$ opposite the side on the boundary of $T$, and the other two sides of $T$, with one side along the side of $R$. For any polygon $X$, let $A(X)$ denote the area of $X$. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where $T$ ranges over all triangles and $R, S$ over all rectangles as above.

1985–A–5 Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx$. For which integers $m$, $1 \leq m \leq 10$ is $I_m \neq 0$?

1986–B–1 Inscribe a rectangle of base $b$ and height $h$ in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of $h$ do the rectangle and triangle have the same area?

1986–B–4 For a positive real number $r$, let $G(r)$ be the minimum value of $|r - \sqrt{m^2 + 2n^2}|$ for all integers $m$ and $n$. Prove or disprove the assertion that $\lim_{r\to\infty} G(r)$ exists and equals 0.

1985–B–5 Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})}\,dt$. You may assume that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

1987–B–1 Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

1986–A–5 Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of $n$ real variables $x = (x_1, \ldots, x_n)$ with continuous second-order partial derivatives everywhere on $\mathbf{R^n}$. Suppose further that there are constants $c_{ij}$ such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all $i$ and $j$, $1 \leq i \leq n$, $1 \leq j \leq n$. Prove that there is a function $g(x)$ on $\mathbf{R^n}$ such that $f_i + \partial g/\partial x_i$ is linear for all $i$, $1 \leq i \leq n$. (A linear function is one of the form

$$a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n.)$$