A Survey of Wavelet Algorithms and Applications, Part 1

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Definition

A Wavelet \( w = w(t) \) is a nice function which is
1. localized in time
2. localized in frequency

and which can be superposed, together with copies of itself produced by transformations like shifts, dilations, or modulations, to produce any desired finite-energy signal.

Example waveforms:
History

- Fourier bases (1822, Paris)
- Gabor functions (1946, *J. IEE*)
- Balian-Low theorem (1981, *CRAS*)
- Wilson bases (1987, Cornell)
- Compactly-supported smooth ortho-normal wavelets (1988, *CPAM*)
- Malvar “LOT” (1990, *IEEE ASSP*)
- Biorthogonal wavelets, wavelet packets, best basis, denoising (1992, *IEEE IT*)
- WSQ fingerprint standard (1993, FBI)
- Local discriminant bases (1994, *CRAS*)
- Multiwavelets (1994, *OE*)
- Wavelets on spheres (1995, *ACM*)
- Sweldens “lifting” (1996, *ACHA*)
- Ridgelets, edgelets, brushlets; spatio-temporal, non-stationary, tight-frame wavelets, ...
Variations on $w_{ab}(t) = w(at + b)$

- continuous indices $a > 0, b$
- discrete indices $a = 2^{-j}, b \in \mathbb{Z}$
- orthonormal $\{w_{jk}\}$
- biorthogonal $\{w_{jk}\}, \{w'_{jk}\}$
- symmetric, antisymmetric
- multidimensional
- matrix dilations $a$
- other parameters $\{w_{abc...}\}$
  - frequency
  - rotation angle
- discrete or finite domain $x$
- adjusted to intervals
- adjusted to curved manifolds
- multiple filters
Difficulties and Solutions

– Few transform standards
  + indexing conventions
  + consistent definitions
  + uniform nomenclature
– Little evaluation beyond small trials
  + use NIST, TIMIT, etc.
  + trade secrets, proprietary information, and patents
  + competition with highly engineered prior art
– Strong mathematical preparation needed
  + new undergraduate courses
  + new graduate programs
Literature

• Bibliographies:
  – http://www.wavelet.org, Wavelet Digest email list. [≈17,000 subscribers]

• Journals:
  – IEEE SP; IT; …
  – SPIE Optical Eng.
  – J. Math. Physics
  – Digital Signal Processing
  – Dr Dobb’s Journal
Actual Analysis
Tilings of the Information Plane

Dirac and Fourier bases.

Narrow and wide window Fourier bases
Tilings (continued...)

Wavelet and wavelet packet bases.

Adapted LOT and general dyadic tiling.
Analysis in Different Tilings

Impulse at increasing Fourier window sizes.

Whistle at increasing Fourier window sizes.
Analysis in Best Bases

Linear and quadratic chirps in best bases.

Superposed chirps in best bases.
Recursive Splitting Algorithms

General Conditions on $H, G$:

- $HH^* = I$ and $GG^* = I$, so $H^*H$ and $G^*G$ are orthogonal projections;
- $HG^* = GH^* = 0$, so $H$ and $G$ project onto independent subspaces;
- $H^*H + G^*G = I$, so $H$ and $G$ together allow perfect reconstruction.
Example: Haar-Walsh splitting

Define

\[ Hx(n) = \frac{[x(2n) + x(2n + 1)]}{2}; \]
\[ Gx(n) = x(2n + 1) - x(2n). \]

\[ H^* x(n) = \begin{cases} 
    x\left(\frac{n}{2}\right), & \text{if } n \text{ is even}; \\
    x\left(\frac{n-1}{2}\right), & \text{if } n \text{ is odd};
\end{cases} \]
\[ G^* x(n) = \begin{cases} 
    -\frac{1}{2}x\left(\frac{n}{2}\right), & \text{if } n \text{ is even}; \\
    \frac{1}{2}x\left(\frac{n-1}{2}\right), & \text{if } n \text{ is odd}. 
\end{cases} \]

Thus

\[ HH^* x(n) = \frac{[H^* x(2n) + H^* x(2n + 1)]}{2} \]
\[ = \frac{[x(n) + x(n)]}{2} = x(n); \]
\[ GG^* x(n) = G^* x(2n + 1) - G^* x(2n) \]
\[ = \frac{1}{2} x(n) - \left[ -\frac{1}{2} x(n) \right] = x(n). \]

\[ x(n) = \begin{cases} 
    Hx\left(\frac{n}{2}\right) - \frac{1}{2}Gx\left(\frac{n}{2}\right), & \text{if } n \text{ even}; \\
    Hx\left(\frac{n-1}{2}\right) + \frac{1}{2}Gx\left(\frac{n-1}{2}\right), & \text{if } n \text{ odd},
\end{cases} \]
\[ = H^* Hx(n) + G^* Gx(n). \]
Discrete Wavelet Transform

Mallat’s original multiresolution algorithm.

...embedded in a wavelet packet decomposition.
Discrete Wavelet Packet Transform

Complete subband, or Walsh-type transform.

Yet another basis in the wavelet packet library.
Underlying Functions

Collett 30

All Haar-Walsh
Best Basis Search

First stage: compute costs, mark leaves.

Middle: mark nodes better than descendents.

Final stage: keep topmost marked nodes.
How Many Graph Bases?

Depth $d$ — $d + 1$ levels — $B_d$ bases.

Recursion $B_d = 1 + B_{d-1}^2 > B_{d-1}^2$, with $B_1 = 2$, implies

$$B_d > 2^{2^d} = 2^N, \quad d > 1.$$
Two-Dimensional Splitting

The operators $H, G$ may be applied separately in more than one variable:

Quadtrees to depth $2^D$.

In $D$ dimensions, each step will produce $2^D$ descendents. Example: face images.

Face minus average face yields caricature.
Application 1: Data Compression

KL: Choose coordinates to concentrate variance.
...with Fast Transforms

Fast KL: Choose only among fast transforms.

Variance in original, fast KL, and KL coordinates.
Joint Best-Basis

Joint best basis (JBB) training algorithm:

1. expand all training images in all bases
2. determine coordinate variances in all bases
3. search for JBB using variance concentration as the cost function
4. keep top few JBB coordinates plus basis description
Good Bases for Images

5-level wavelet basis, used in JPEG-2000.

5-level wavelet packet basis, used in WSQ.
Application 2: Classification

Problem: Given a training set divided into two classes A and B, find a basis of wavelets that maximizes a discriminant function.

Left: Wavelet from JBB. Right: KL eigenface.

Advantages of wavelet features:
- nice basis functions
- fast pre-processing transforms

Difficulties with wavelet features:
- no shift invariance
- classifying features are non-intuitive
Local discriminant basis (LDB) training:
1. expand all of both classes in all bases
2. determine coordinate discrimination power in all bases
3. search for LDB using discrimination power concentration as the cost function
4. keep top few LDB coordinates plus basis description