Some Theory,
But More Applications
of Wavelet Packets

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Definition

A *Wavelet* $w = w(t)$ is a nice function which is

1. localized in time
2. localized in frequency

and which can be superposed, together with copies of itself produced by transformations like shifts, dilations, or modulations, to produce any desired finite-energy signal.
Example waveforms
History

- Fourier bases (1822, Paris)
- Gabor functions (1946, *J. IEE*)
- Balian-Low theorem (1981, *CRAS*)
- Wilson bases (1987, Cornell)
- Compactly-supported smooth ortho-normal wavelets (1988, *CPAM*)
- Malvar “LOT” (1990, *IEEE ASSP*)
- Biorthogonal wavelets, wavelet packets, best basis, denoising (1992, *IEEE IT*)
- WSQ fingerprint standard (1993, FBI)
- Local discriminant bases (1994, *CRAS*)
- Multiwavelets (1994, *OE*)
- Wavelets on spheres (1995, *ACM*)
- Sweldens “lifting” (1996, *ACHA*)
- Ridgelets, edgelets, brushlets; spatio-temporal, non-stationary, tight-frame wavelets,…
Variations on $w_{ab}(t) = w(at + b)$

- continuous indices $a > 0$, $b$
- discrete indices $a = 2^{-j}$, $b \in \mathbb{Z}$
- orthonormal $\{w_{jk}\}$
- biorthogonal $\{w_{jk}\}, \{w'_{jk}\}$
- symmetric, antisymmetric
- multidimensional
- matrix dilations $a$
- other parameters $\{w_{abc...}\}$
  - frequency
  - rotation angle
- discrete or finite domain $x$
- adjusted to intervals
- adjusted to curved manifolds
- multiple filters
Difficulties and Solutions

- Few transform standards
  + indexing conventions
  + consistent definitions
  + uniform nomenclature
- Little evaluation beyond small trials
  + use NIST, TIMIT, etc.
  + trade secrets, proprietary information, and patents
  + competition with highly engineered prior art
- Strong mathematical preparation needed
  + new undergraduate courses
  + new graduate programs
Literature

• Bibliographies:
  – [http://www.wavelet.org](http://www.wavelet.org), Wavelet Digest email list. [~17,000 subscribers]

• Journals:
  – IEEE SP; IT; …
  – SPIE Optical Eng.
  – J. Math. Physics
  – Digital Signal Processing
  – Dr Dobb’s Journal
Actual Analysis
Dirac and Fourier Information Plane Tilings
STFT Tilings of the Information Plane

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Wavelet and Wavelet Packet Tilings
Adapted LOT and General Dyadic Tilings
Impulse Analysis in Different Tilings

Impulse at increasing Fourier window sizes.
Tone Analysis in Different Tilings

Whistle at increasing Fourier window sizes.
Chirp Analysis in Best Bases
Superposed Chirp Analysis in Different Tilings
Recursive Splitting Algorithms

General Conditions on $H, G$:

- $HH^* = I$ and $GG^* = I$, so $H^*H$ and $G^*G$ are orthogonal projections;
- $HG^* = GH^* = 0$, so $H$ and $G$ project onto independent subspaces;
- $H^*H + G^*G = I$, so $H$ and $G$ together allow perfect reconstruction.
Example: Haar-Walsh splitting

Define

\[ Hx(n) = \frac{[x(2n) + x(2n + 1)]}{2}; \]
\[ Gx(n) = x(2n + 1) - x(2n). \]

\[ H^*x(n) = \begin{cases} 
  x(\frac{n}{2}), & \text{if } n \text{ is even;} \\
  x(\frac{n-1}{2}), & \text{if } n \text{ is odd;}
\end{cases} \]
\[ G^*x(n) = \begin{cases} 
  -\frac{1}{2} x(\frac{n}{2}), & \text{if } n \text{ is even;} \\
  \frac{1}{2} x(\frac{n-1}{2}), & \text{if } n \text{ is odd.}
\end{cases} \]

Thus

\[ HH^*x(n) = \frac{[H^*x(2n) + H^*x(2n + 1)]}{2} \]
\[ = [x(n) + x(n)]/2 = x(n); \]
\[ GG^*x(n) = G^*x(2n + 1) - G^*x(2n) \]
\[ = \frac{1}{2} x(n) - [-\frac{1}{2} x(n)] = x(n). \]

\[ x(n) = \begin{cases} 
  Hx(\frac{n}{2}) - \frac{1}{2} Gx(\frac{n}{2}), & \text{if } n \text{ is even;} \\
  Hx(\frac{n-1}{2}) + \frac{1}{2} Gx(\frac{n-1}{2}), & \text{if } n \text{ is odd,}
\end{cases} \]
\[ = H^*Hx(n) + G^*Gx(n). \]
Discrete Wavelet Transform (DWT) 1

Mallat’s original multi-resolution DWT...
Discrete Wavelet Transform 2

... embedded in a discrete wavelet packet decomposition.
Discrete Wavelet Packet Transform 1

Complete subband, or Walsh-type transform.
Discrete Wavelet Packet Transform 2

Yet another basis in the wavelet packet library.
Underlying Functions

All Haar-Walsh

Coiflet 30
Best Basis Search 1

First stage: compute costs, mark leaves.
Best Basis Search 2

Middle: mark nodes better than descendents.
Best Basis Search 3

Final stage: keep topmost marked nodes.
How Many Graph Bases 1

\[ N = 2^d \]

Depth \( d \) — \( d + 1 \) levels — \( B_d \) bases.
How Many Graph Bases 2

Recursion $B_d = 1 + B_{d-1}^2 > B_{d-1}^2$, with $B_1 = 2$, implies

$$B_d > 2^{2^d} = 2^N, \quad d > 1.$$
Two-Dimensional Splitting

Apply operators $H, G$ separately for multidimensions:

Dimension 2: quadtree to depth 2.

In $D$ dimensions, each step will produce $2^D$ descendents.
Example: Face Images

Face minus average face yields caricature.
Application 1: Data Compression

KL: Choose coordinates to concentrate variance.
Fast KL

Choose only among low-complexity transforms.
Fast Transforms Work Pretty Well

Variance in original, fast KL, and KL coordinates.
Joint Best-Basis

Joint best basis (JBB) training algorithm:
1. expand all training images in all bases
2. determine coordinate variances in all bases
3. search for JBB using variance concentration cost function
4. keep top few JBB coordinates plus basis description
Good Bases for Images 1

5-level wavelet basis, used in JPEG-2000.
Good Bases for Images 2

5-level wavelet packet basis, used in WSQ.
Application 2: Classification

PROBLEM: Given a training set divided into two classes A and B, find a wavelet packet basis that maximizes a discriminant function.

Left: Wavelet from JBB. Right: KL eigenface.
Wavelet Features for Classification

Advantages of wavelet features:
• nice basis functions
• fast pre-processing transforms

Difficulties with wavelet features:
• no shift invariance
• classifying features are non-intuitive
Local Discriminant Basis

Local discriminant basis (LDB) training:
1. expand all of both classes in all bases
2. determine coordinate discrimination power in all bases
3. search for LDB using discrimination power concentration as the cost function
4. keep top few LDB coordinates plus basis description