Vanishing of Hyperelliptic L-Functions at the Central Point

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In the book "The Riemann Hypothesis and Hilbert's Tenth Problem", Chowla raised the following conjecture.

Conjecture (Chowla, 1965)

For any quadratic Dirichlet character χ, $L(s, \chi) \neq 0$ for all $s \in (0, 1)$.

In particular, it suggests $L(1/2, \chi) \neq 0$.

Theorem (Soundrarajan, 2000)

At least 87.5% of odd squarefree integers $d > 0$ have the property that $L(1/2, \chi_{8d}) \neq 0$ where χ_{8d} denotes the real quadratic character with conductor $8d$.

Function Field

Number field	Function field
$\mathbb{Q}$	$\mathbb{F}_q(x)$
$\mathbb{Z}$	$\mathbb{F}_q[x]$
positive primes	monic, irreducible polynomials
$\lvert n \rvert$	$\lvert f \rvert = q^{\deg f}$

Let $D \in \mathbb{F}_q[x]$ be monic and squarefree. Then we define a quadratic character χ_D as follows.

For P a prime in $\mathbb{F}_q(x)$,

$$\chi_D(P) = \begin{cases} 1 & \text{P splits in } \mathbb{F}_q(x)(\sqrt{D}) \\ -1 & \text{P is inert in } \mathbb{F}_q(x)(\sqrt{D}) \\ 0 & \text{P ramifies in } \mathbb{F}_q(x)(\sqrt{D}) \end{cases}$$

Function field

Definition

Let $\mathbb{F}_q$ be a finite field with odd characteristic and let

$$g(N) = \{D \in \mathbb{F}_q[x], \text{ monic, squarefree} : |D| < N, L(1/2, \chi_D) = 0\}$$

Question: Is $g(N)$ equal to $\emptyset$?

Theorem (Bui–Florea, 2016)

With the notation above,

$$|g(N)| \ll 0.057N + o(N)$$

for any $N = q^{2n+1}$ where $n \in \mathbb{Z}$.

Main theorem

Theorem (L., 2017)

When q is a square, for any $\epsilon > 0$,

$$|g(N)| \geq B_\epsilon N^{1/2-\epsilon}$$

with some nonzero constant B_ϵ and $N > N_\epsilon$.

Although the analogous statement of Chowla's conjecture does not hold over $\mathbb{F}_q(x)$, it may hold for almost all quadratic characters, i.e. it may be the case that $|g(N)|/N \to 0$ as $N \to \infty$.

Geometric Interpretation

Let $D \in \mathbb{F}_q[x]$ be a monic, squarefree polynomial. Over $\mathbb{F}_q$, it defines a hyperelliptic curve

$$C : y^2 = D(x).$$

Let $P(x) \in \mathbb{Z}[x]$ be the characteristic polynomial of geometric Frobenius acting on the Jacobian $J(C)$.

Then,

$$\begin{aligned} L(1/2, \chi_D) = 0 &\iff P(q^{-1/2}) = 0 \\ &\iff \sqrt{q} \text{ is a Frobenius eigenvalue} \end{aligned}$$

Geometric Interpretation

By Honda–Tate theory, when q is a square, there exists an elliptic curve E_0 over $\mathbb{F}_q$ which admits $\sqrt{q}$ as a Frobenius eigenvalue. Moreover, any abelian variety with $\sqrt{q}$ being a Frobenius eigenvalue has E_0 as an isogenous factor.

Thus,

$$L(1/2, \chi_D) = 0 \iff P(q^{-1/2}) = 0 \iff J(C) \sim E_0 \times A$$

for some abelian variety A.

Moreover,

$$J(C) \sim E_0 \times A \iff \exists \text{ dominant map, } C \to E_0$$

Maps Between Hyperelliptic Curves

Proposition (L., 2017)

Let C_0 be a hyperelliptic curve defined over $\mathbb{F}_q$ with an odd degree defining equation or an even degree defining equation of the form $y^2 = f$ where f is reducible.
For any $\epsilon > 0$, there exist positive constants B_ϵ and N_ϵ such that the number of polynomials $D \in \mathbb{F}_q[x]$ satisfying

- $|D| < N$
- $C : y^2 = D$ *admits a dominant map to* C_0

is at least $B_\epsilon \cdot N^{\frac{1}{g+1} - \epsilon}$ *for* $N > N_\epsilon$.

Application to Ranks of Elliptic Curves

From $E_0 : Y^2 = f(X)$ over $\mathbb{F}_q$, we construct the constant elliptic curve over the rational function field $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(x)$.

Let $C : y^2 = D(x)$ be a hyperelliptic curve over $\mathbb{F}_q$.

$$\exists \text{ dominant map, } C \to E_0 \iff \operatorname{rank} E_D > 0$$

where E_D is the quadratic twist of E by D.

$$\begin{tikzcd} C \arrow[r] \arrow[d, "{(h(x),p(x)y)}"'] & \mathbb{P}^1 \arrow[d, "h"] \\ E_0 \arrow[r, "{(X,Y)\to X}"'] & \mathbb{P}^1 \end{tikzcd}$$

Since we have $y^2 = D$ and $p^2(x)y^2 = f(h(x))$, the point with coordinate $(h(x), p(x))$ lies on $DY^2 = f(X)$ over $\mathbb{F}_q(x)$.

Application to Ranks of Elliptic Curves

Corollary (L., 2017)

Let $E = E_0 \times \mathbb{F}_q(x)$ *be a constant elliptic curve over* $\mathbb{F}_q(x)$.
Let $R_m(N) = \{D \in \mathbb{F}_q[x] : \textit{monic, squarefree, } |D| < N, \operatorname{rank} E_D \geq m\}$.
Then for any $\epsilon > 0$,

$$|R_2(N)| \gg N^{1/2-\epsilon}$$

Corollary (L., 2017)

Let $E/\mathbb{F}_q$ *be an elliptic curve with* $\sqrt{q}$ *as a Frobenius eigenvalue, define*

$$P(g) = \{D \in \mathbb{F}_q[x] : \textit{ monic, squarefree, of odd degree, } \deg D \leq 2g+1\},$$

$$R(g) = \{D \in P'(g) : E_D \textit{ has rank } 0\}.$$

Then

$$\lim_{g \to \infty} \frac{|R(g)|}{|P(g)|} \geq 0.9427\cdots + o(1).$$

Data

$\mathbb{F}_9$			
Degree d	$\lvert g'(9^d)\rvert$	$9^d - 9^{d-1}$	$\frac{log(\lvert g'(9^d)\rvert)}{log(9^d - 9^{d-1})}$
3	6	648	0.2768
4	18	5832	0.3333
5	216	52488	0.4946
6	180	472392	0.3975
7	8658	4251528	0.5940

For degree 8, 9 and 10, due to the large number of monic squarefree polynomials, we randomly sampled 5000000 data points for each and got the following data. The sample set is denoted by S.

Degree d	$\lvert S \cap g'(9^d)\rvert$	$\lvert S\rvert$	$\frac{log(\lvert g'(9^d)\rvert)}{log(9^d - 9^{d-1})}$
8	2660	5000000	0.5682
9	3262	5000000	0.6269
10	532	5000000	0.5814

Data

$\mathbb{F}_5$			
Degree d	$\lvert g'(5^d)\rvert$	$5^d - 5^{d-1}$	$\frac{log(\lvert g'(5^d)\rvert)}{log(5^d - 5^{d-1})}$
3	0	100	
4	0	500	
5	1	2500	0
6	0	12500	
7	10	62500	0.2085
8	5	312500	0.1272

For degree 9 and 10, similarly, we sampled 5000000 data points for each. The sample set is again denoted by S.

Degree d	$\lvert S \cap g'(5^d)\rvert$	$\lvert S\rvert$	$\frac{log(\lvert g'(5^d)\rvert)}{log(5^d - 5^{d-1})}$
9	317	5000000	0.3222
10	89	5000000	0.3109

References

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