Effective Bounds on the Dimensions of Jacobians Covering Abelian Varieties

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Abelian Varieties as Quotients of Jacobians

Let $A \subset \mathbb{P}^r$ be an abelian variety and C be a smooth curve given by the instersection of A and hyperplanes $H_i \subset \mathbb{P}^r$. By the universal property of Jacobians, one has the following diagram:

$$\begin{array}{ccc} C = A \cap (\cap H_i) & \xrightarrow{\iota} & \mathrm{Jac}(C) \\ & \searrow & \Big\downarrow{\scriptstyle \pi} \\ & & A \hookrightarrow \mathbb{P}^r \end{array}$$

where $\iota : C \to \mathrm{Jac}(C)$ is the Abel-Jacobi map for C.

Since C is a complete intersection on A, the map π is dominant.

Abelian Varieties as Quotients of Jacobians

Combined with Bertini's theorem, when the base field is infinite, every principally polarized abelian variety is covered by the Jacobian variety of a smooth curve.

Moreover, one can give an upper bound on the genus of C.

$$\dim Jac(C) \leq \left\lfloor \frac{\deg(A)-1}{r-1} \right\rfloor \left(\deg(A) - \frac{\left\lfloor \frac{\deg(A)-1}{r-1} \right\rfloor + 1}{2}(r-1) - 1 \right).$$

Question

What happens over finite fields?

Poonen's Bertini's Theorem Over Finite Fields

Theorem (Poonen, 2004)

Let X be a smooth quasi-projective subscheme of $\mathbb{P}^n$ of dimension $m \geq 0$ over $\mathbb{F}_q$. Then there exist homogeneous polynomials f over $\mathbb{F}_q$ for which the intersection of X and the hypersurface $f = 0$ is smooth.

Poonen's Bertini's Theorem Over Finite Fields

Theorem (Bucur–Kedlaya, 2012)

Let $X \subset \mathbb{P}^r_{\mathbb{F}_q}$ be a smooth projective scheme of dimension $n \geq 0$ over a finite field $\mathbb{F}_q$ of characteristic p. Choose an integer $k \in \{1, \ldots, n-1\}$, a degree sequence $d = (d_1 \leq d_2 \leq \cdots \leq d_k)$, and set

$$\mathcal{P}_d = \left\{ f \in S_d \;\middle|\; \begin{array}{c} X \cap V(f) \text{ has dimension } n-k \\ \text{and is smooth} \end{array} \right\}.$$

Then

$$\left| \frac{\#\mathcal{P}_d}{\#S_d} - \text{Main term independent of } d \right| \leq \text{Error term decreasing with } d$$

Main Theorem

Theorem (Bruce–L., 2018)

Fix $r, n \in \mathbb{N}$ and let $\mathbb{F}_q$ be a field of characteristic p. There exists an explicit constant $C_{n,r,q}$ such that if $A \subset \mathbb{P}^r_{\mathbb{F}_q}$ is a principally polarized abelian variety of dimension n, then for any $d \in \mathbb{N}$ satisfying

$$C_{n,r,q}\zeta_A\left(n+\frac{1}{2}\right)\deg(A) \leq \frac{q^{\frac{d}{\max\{n+1,p\}}}(d+1)}{d^{n+1}+d^n+q^{\frac{d}{\max\{n+1,p\}}}},$$

there exists a smooth curve over $\mathbb{F}_q$ whose Jacobian J maps dominantly onto A, where

$$\dim J \leq \left\lfloor \frac{\deg(A)d^{n-1}-1}{r-1} \right\rfloor \left(\deg(A)d^{n-1} - \frac{\left\lfloor \frac{\deg(A)d^{n-1}-1}{r-1} \right\rfloor + 1}{2}(r-1) - 1\right).$$

When A is Simple

If $\phi_1 : C_1 \to A$ is a nonconstant map, then we construct a map $\phi_2 : C_2 \to A$ such that C_2 is a smooth curve, ϕ_2 factors through ϕ_1, and the induced map $\mathrm{Jac}(C_2) \to A$ is dominant.

$$C_2 = \tilde{C}'_{1,red} \xrightarrow{\pi} C'_{1,red} \longrightarrow C_1 \xrightarrow{\phi_1} A, \qquad \phi_2 : \tilde{C}'_{1,red} \to A$$

- C_1 : connected curve, complete intersection in A
- $C'_{1,red}$: reduced, irreducible component of C_1
- $\tilde{C}'_{1,red}$: normalization of $C'_{1,red}$

$$g\left(\tilde{C}'_{1,red}\right) = p_a\left(\tilde{C}'_{1,red}\right) \leq p_a\left(C'_{1,red}\right) \leq \deg(A)^2 d^{2n-2} + \deg(A) d^{n-1} - 2.$$

When A is Simple

Theorem (Bruce–L., 2018)

If $A \subset \mathbb{P}^r_{\mathbb{F}_q}$ is a simple abelian variety, then for any $d \in \mathbb{N}$ satisfying

$$\deg(A) \leq \frac{(d-1)q^{\frac{1}{2}(d+1)(d+2)}}{d^{n-1}-1},$$

there exists a smooth curve over $\mathbb{F}_q$ whose Jacobian J maps dominantly onto A, where

$$\dim J \leq \deg(A)d^{n-1}\left(\deg(A)d^{n-1}+1\right).$$

Application

Corollary

Fix a finite field $\mathbb{F}_q$ of characteristic p and $n \in \mathbb{N}$. There exists a constant $C_{n,q}$ such that if E is an elliptic curve over $\mathbb{F}_q$ then there exists a smooth curve C of genus $g \leq C_{n,q}$ defined over $\mathbb{F}_q$ such that $\mathrm{Jac}(C)$ admits E^n as an isogeny factor, i.e.

$$Jac(C) \sim E^n \times A$$

for some abelian variety A.

References

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[4] Bjorn Poonen, *Bertini theorems over finite fields*, Ann. of Math. (2) **160** (2004), no. 3, 1099–1127.