The Essential Norm of Operators on the Bergman Space

Brett D. Wick

Georgia Institute of Technology
School of Mathematics

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This talk is based on joint work with:

Daniel Suárez  
University of Buenos Aires

Mishko Mitkovski  
Clemson University
Weighted Bergman Spaces on $\mathbb{B}_n$

- Let $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$.
- For $\alpha > -1$, we let
  \[ dv_\alpha(z) := c_\alpha (1 - |z|^2)^\alpha \, dv(z), \quad \text{with} \quad c_\alpha := \frac{\Gamma(n + \alpha + 1)}{n! \, \Gamma(\alpha + 1)}. \]
  The choice of $c_\alpha$ gives that $v_\alpha(\mathbb{B}_n) = 1$.
- For $1 < p < \infty$ the space $A^p_\alpha$ is the collection of holomorphic functions on $\mathbb{B}_n$ such that
  \[ \|f\|_{A^p_\alpha}^p := \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) < \infty. \]
- For $\lambda \in \mathbb{B}_n$ let $k^{(p,\alpha)}_\lambda(z) = \frac{(1-|\lambda|^2)^{n+1+\alpha}}{(1-\lambda^2z)^{n+1+\alpha}}$ and $K_\lambda(z) = \frac{1}{(1-\lambda^2z)^{n+1+\alpha}}$.
- A computation shows: $\left\| k^{(p,\alpha)}_\lambda \right\|_{A^p_\alpha} \approx 1$. 
Weighted Bergman Spaces on $\mathbb{D}^n$

- Let $\mathbb{D}^n := \{ z \in \mathbb{C}^n : |z_l| < 1 \quad l = 1, \ldots, n \}$.

- We let
  \[ dv_\alpha(z) := \frac{1}{\pi^n} dA(z_1) \cdots dA(z_n) \]

- For $1 < p < \infty$ the space $A^p$ is the collection of holomorphic functions on $\mathbb{D}^n$ such that
  \[ \|f\|^p_{A^p(\mathbb{D}^n)} := \int_{\mathbb{D}^n} |f(z)|^p dv(z) < \infty. \]

- For $\lambda \in \mathbb{D}^n$ let
  \[ k_\lambda^{(p)} (z) := \prod_{l=1}^n \frac{(1-|\lambda_l|^2)^{\frac{2}{q}}}{(1-\bar{\lambda}_lz_l)^2} \]
  and
  \[ K_\lambda(z) = \prod_{l=1}^n \frac{1}{(1-\lambda_lz_l)^2}. \]

- A computation shows: $\left\| k_\lambda^{(p)} \right\|_{A^p(\mathbb{D}^n)} \approx 1.$
Toeplitz Operators and the Toeplitz Algebra

- The projection of $L^2$ onto $A^2$ is given by the integral operator
  \[ P(f)(z) := \int_{\Omega} f(w)K_w(z) \, dv(w). \]

- This operator is bounded from $L^p$ to $A^p$ when $1 < p < \infty$.

- Let $M_a$ denote the operator of multiplication by the function $a$, $M_a(f) := af$. The Toeplitz operator with symbol $a \in L^\infty$ is the operator given by
  \[ T_a := PM_a. \]

- It is immediate to see that $\|T_a\|_{L(A^p)} \lesssim \|a\|_{L^\infty}$.

- More generally, for a measure $\mu$ we will define the operator
  \[ T_\mu f(z) := \int_{\Omega} f(w)K_w(z) \, d\mu(w), \]
  which will define an analytic function for all $f \in H^\infty$. 

Motivations for the Problem

Toeplitz Operators and the Toeplitz Algebra

- For symbols in $L^\infty$ we let $\mathcal{T}_p$ be the $C^*$ subalgebra of $\mathcal{L}(A^p)$ generated by $T_a$.
- An important class of operators in $\mathcal{T}_p$ are those that are finite sums of finite products of Toeplitz operators. Namely, for symbols $a_{jk} \in L^\infty$ with $1 \leq j \leq J$ and $1 \leq k \leq K$ we will need to study the operators:

$$\sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}$$

- Additionally,

$$\mathcal{T}_p = \left\{ \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}} : a_{jk} \in L^\infty, \ 1 \leq j \leq J, \ 1 \leq k \leq K \right\} \mathcal{L}(A^p)$$
Geometry of the Domain $\Omega$

When $\Omega = \mathbb{B}_n$ let

$$d\lambda(z) := \frac{dv(z)}{(1 - |z|^2)^{n+1}}$$

and when $\Omega = \mathbb{D}_n$ let

$$d\lambda(z) := \frac{dv(z)}{\prod_{l=1}^n (1 - |z_l|^2)^2}$$

Lemma (Geometric Decomposition of $\Omega$)

There exists an integer $N > 0$ (depending only on the doubling constant of the measure $\lambda$) such that for any $r > 0$ there is a covering $\mathcal{F}_r = \{F_j\}$ of $\Omega$ by disjoint Borel sets satisfying

1. every point of $\Omega$ belongs to at most $N$ of the sets $G_j := \{z \in \Omega : d(z, F_j) \leq r\}$,
2. $\text{diam}_\beta F_j \leq 2r$ for every $j$. 

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Motivations for the Problem

Carleson Measures for $A^p_\alpha(B_n)$

A measure $\mu$ on $B_n$ is a Carleson measure for $A^p_\alpha$ if

$$\int_{B_n} |f(z)|^p \, d\mu(z) \lesssim \int_{B_n} |f(z)|^p \, dv_\alpha(z) \quad \forall f \in A^p_\alpha.$$ 

Lemma (Characterizations of $A^p_\alpha(B_n)$ Carleson Measures)

Suppose that $1 < p < \infty$ and $\alpha > -1$. Let $\mu$ be a measure on $B_n$ and $r > 0$. The following quantities are equivalent, with constants that depend on $n$, $\alpha$ and $r$:

1. $\|\mu\|_{\text{RKM}} := \sup_{z \in B_n} \int_{B_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\bar{z}w|^{2(n+1+\alpha)}} \, d\mu(w)$;

2. $\|\nu_r\|^p := \inf \left\{ C : \int_{B_n} |f(z)|^p \, d\mu(z) \leq C \int_{B_n} |f(z)|^p \, dv_\alpha(z) \right\}$;

3. $\|\mu\|_{\text{Geo}} := \sup_{z \in B_n} \frac{\mu(D(z,r))}{(1-|z|^2)^{n+1+\alpha}}$;

4. $\|T_\mu\|_{\mathcal{L}(A^p_\alpha(B_n))}$.
Carleson Measures for $A^p(\mathbb{D}^n)$

A measure $\mu$ on $\mathbb{D}^n$ is a Carleson measure for $A^p_\alpha$ if

$$\int_{\mathbb{D}^n} |f(z)|^p \, d\mu(z) \lesssim \int_{\mathbb{D}^n} |f(z)|^p \, dv(z) \quad \forall f \in A^p.$$

Lemma (Characterizations of $A^p(\mathbb{D}^n)$ Carleson Measures)

Suppose that $1 < p < \infty$. Let $\mu$ be a measure on $\mathbb{D}^n$ and $r > 0$. The following quantities are equivalent, with constants that depend on $n$, $\alpha$ and $r$:

1. $\|\mu\|_{\text{RKM}} := \sup_{z \in \mathbb{D}^n} \int_{\mathbb{D}^n} \prod_{l=1}^n \left( \frac{(1-|z_l|^2)^2}{|1-\overline{z_l}w_l|^4} \right) d\mu(w)$;

2. $\|i_p\|^p := \inf \left\{ C : \int_{\mathbb{D}^n} |f(z)|^p \, d\mu(z) \leq C \int_{\mathbb{D}^n} |f(z)|^p \, dv(z) \right\}$;

3. $\|\mu\|_{\text{Geo}} = \sup_{z \in \mathbb{D}^n} \frac{\mu(D(z,r))}{\prod_{l=1}^n (1-|z_l|^2)^2} \approx \sup_{z \in \mathbb{D}^n} \frac{\mu(D(z,r))}{v(D(z,r))}$;

4. $\|T_\mu\|_{\mathcal{L}(A^p(\mathbb{D}^n))}$. 
The Berezin Transform

For $S \in \mathcal{L}(A^p)$, we define the Berezin transform by

$$B(S)(z) := \left\langle S k_z^{(p)}, k_z^{(q)} \right\rangle_{A^2}.$$

- $B : \mathcal{L}(A^p) \to L^\infty(\Omega)$.
- If $S$ is compact, then $B(S)(z) \to 0$ as $z \to \partial \Omega$.
- The Berezin transform is one-to-one.
- $B(S)$ is Lipschitz continuous with respect to the hyperbolic metric

$$|B(S)(z_1) - B(S)(z_2)| \leq \sqrt{2} \|S\|_{\mathcal{L}(A^p)} \beta(z_1, z_2)$$

- Range of $B$ is not closed: $B^{-1} : B(\mathcal{L}(A^p)) \to \mathcal{L}(A^p)$ is not bounded.
Motivations for the Problem

Related Results


Suppose that \( a_{jk} \in L^\infty(\mathbb{D}) \) with \( 1 \leq j \leq J \) and \( 1 \leq k \leq K \). Let \( S = \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}} \). The following are equivalent:

(a) The operator \( S \) is compact on \( A^2(\mathbb{D}) \);

(b) \( B(S)(z) \to 0 \) as \( |z| \to 1 \);

(c) \( \|S_k z\|_{A^2} \to 0 \) as \( |z| \to 1 \).

- The interesting implication is \((b) \implies (a)\);
- The same proof works in the case of the unit ball, but was done by Raimondo. Also works in the polydisc by Nam and Zheng.

**Theorem (Engliš, Ark. Mat. 30 (1992))**

Let \( 1 < p < \infty \). If \( S \) is a compact operator on \( A^p \), then \( S \in \mathcal{T}_p \).
Main Question of Interest

From the previous Theorem and simple functional analysis we have that if $S$ is compact on $A^p$ then

$$S \in \mathcal{T}_p \quad \text{and} \quad B(S)(z) \to 0 \text{ as } z \to \partial \Omega.$$ 

Question (Characterizing the Compacts)

If $S \in \mathcal{T}_p$ and $B(S)(z) \to 0$ as $z \to \partial \Omega$, then is $S$ is compact?

Yes!

- Shown to be true by Suárez for $A^p$ when $1 < p < \infty$ and $\alpha = 0$.
- Extended to $\alpha > -1$ by Suárez, Mitkovski and BDW.
- Extended to the polydisc $\mathbb{D}^n$ by Mitkovski and BDW.
Characterizations of Compactness and Essential Norm

Theorem

Let $1 < p < \infty$ and $S \in \mathcal{L}(A^p)$. Then $S$ is compact if and only if $S \in \mathcal{T}_p$ and $\lim_{z \to \partial \Omega} B(S)(z) = 0$.

We can actually obtain much more precise information about the essential norm of an operator. For $S \in \mathcal{L}(A^p_\alpha)$ recall that

$$\|S\|_e = \inf \left\{ \|S - Q\|_{\mathcal{L}(A^p_\alpha)} : Q \text{ is compact} \right\}.$$ 

Two other measures of the “size” of an operator $S \in \mathcal{L}(A^p)$:

$$b_S := \sup_{r > 0} \limsup_{z \to \partial \Omega} \left\| M_{1D(z,r)} S \right\|_{\mathcal{L}(A^p,L^p)}$$

$$c_S := \lim_{r \to 1} \left\| M_{1\Omega \setminus r\Omega} S \right\|_{\mathcal{L}(A^p,L^p)}.$$
Characterizations of Compactness and Essential Norm

Let $r > 0$ and let $\{w_m\}$ and $D_m$ be the sets that form a lattice in $\Omega$. Define the measure

$$\mu_r = \sum_{m} v(D_m) \delta_{w_m}.$$ 

It is well known that $\mu_r$ is a $A^p$ Carleson measure, so $T_{\mu_r} : A^p \to A^p$ is bounded.

**Lemma**

$T_{\mu_r} \to \text{Id}$ on $L(A^p)$ when $r \to 0$.

Let $r > 0$ be chosen so that $\| T_{\mu_r} - \text{Id} \|_{L(A^p)} < \frac{1}{4}$, and $\mu := \mu_r$. Then set

$$a_S(\rho) := \limsup_{z \to \partial \Omega} \sup \left\{ \| Sf \|_{A^p} : f \in T_{\mu_1 D(z, \rho)} (A^p), \| f \|_{A^p} \leq 1 \right\}$$

and define

$$a_S := \lim_{\rho \to 1} a_S(\rho).$$
Main Results

Characterizations of Compactness and Essential Norm

Theorem

Let $1 < p < \infty$ and let $S \in \mathcal{T}_p$. Then there exists constants depending only on $n$ and $p$ such that:

$$a_S \approx b_S \approx c_S \approx \|S\|_e.$$

For the automorphism $\varphi_z$ such that $\varphi_z(0) = z$ define the map

$$U_z^{(p)} f(w) := f(\varphi_z(w)) \frac{(1 - |z|^2)^{n+1+\alpha}}{(1 - w\overline{z})^{2(n+1+\alpha)}}.$$

A standard change of variable argument and computation gives that

$$\left\| U_z^{(p)} f \right\|_{A^p} = \|f\|_{A^p} \quad \forall f \in A^p.$$
Characterizations of Compactness and Essential Norm

For $z \in \Omega$ and $S \in \mathcal{L}(A^p)$ we then define the map

$$S_z := U_z^{(p, \alpha)} S(U_z^{(q, \alpha)})^*.$$ 

One should think of the map $S_z$ in the following way. This is an operator on $A^p$ and so it first acts as “translation” in $\Omega$, then the action of $S$, then “translation” back.

**Theorem**

*Let $1 < p < \infty$ and $S \in \mathcal{T}_p$. Then*

$$\| S \|_e \approx \sup_{\| f \|_{A^p} = 1} \limsup_{z \to \partial \Omega} \| S_z f \|_{A^p}.$$
Sketch of Proofs

Main Ingredients

Connecting the Geometry and Operator Theory

Lemma

Let $S \in \mathcal{T}_p$, $\mu$ a Carleson measure and $\epsilon > 0$. Then there are Borel sets $F_j \subset G_j \subset \Omega$ such that

(i) $\Omega = \bigcup F_j$;

(ii) $F_j \cap F_k = \emptyset$ if $j \neq k$;

(iii) each point of $\Omega$ lies in no more than $N(n)$ of the sets $G_j$;

(iv) $\text{diam}_\beta G_j \leq d(p, S, \epsilon)$

and

$$\left\| ST_\mu - \sum_{j=1}^{\infty} M_{1_{F_j}} ST_{1_{G_j}} \mu \right\|_{\mathcal{L}(A^p, L^p)} < \epsilon.$$
A Uniform Algebra and its Maximal Ideal Space

- Let $\mathcal{A}$ denote the bounded functions that are uniformly continuous from the metric space $(\Omega, \beta)$ into the metric space $(\mathbb{C}, |\cdot|)$.
- Associate to $\mathcal{A}$ its maximal ideal space $M_\mathcal{A}$ which is the set of all non-zero multiplicative linear functionals from $\mathcal{A}$ to $\mathbb{C}$.
- Since $\mathcal{A}$ is a $C^*$ algebra we have that $\Omega$ is dense in $M_\mathcal{A}$.
- The Toeplitz operators associated to symbols in $\mathcal{A}$ are useful to study the Toeplitz algebra $T_p$.

**Theorem**

The Toeplitz algebra $T_p$ equals the closed algebra generated by $\{T_a : a \in \mathcal{A}\}$. 
A Uniform Algebra and its Maximal Ideal Space

- For an element \( x \in M_A \setminus \Omega \) choose a net \( z_\omega \to x \).
- Form \( S_{z_\omega} \) and look at the limit operator obtained when \( z_\omega \to x \), denote it by \( S_x \).

**Lemma**

Let \( S \in \mathcal{L}(A^p) \). Then \( B(S)(z) \to 0 \) as \( z \to \partial \Omega \) if and only if \( S_x = 0 \) for all \( x \in M_A \setminus \Omega \).

We can extend this to compute the essential norm of an operator \( S \) in terms of \( S_x \) where \( x \in M_A \setminus \Omega \).

**Theorem**

Let \( S \in \mathcal{T}_p \). Then there exists a constant \( C(p, n) \) such that

\[
\sup_{x \in M_A \setminus \Omega} \| S_x \|_{\mathcal{L}(A^p)} \approx \| S \|_e.
\]
Proof of Main Theorem

Theorem

Let $1 < p < \infty$ and $S \in \mathcal{L}(A^p)$. Then $S$ is compact if and only if $S \in \mathcal{T}_p$ and $\lim_{z \to \partial \Omega} B(S)(z) = 0$.

Proof.

$\Rightarrow$: If $S$ is compact that $B(S)(z) \to 0$ as $z \to \partial \Omega$ and $S \in \mathcal{T}_p$.

$\Leftarrow$: If $S \in \mathcal{T}_p$, then we have

$$\sup_{x \in M_A \setminus \Omega} \|S_x\|_{\mathcal{L}(A^p)} \approx \|S\|_e.$$  

If $B(S)(z) \to 0$ as $z \to \partial \Omega$, then $S_x = 0$ for all $x \in M_A \setminus \Omega$. This gives $\|S\|_e = 0$ or equivalently $S$ is compact. \hfill $\square$
Thank You!