Composition of Haar Paraproducts

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Sarason’s Conjecture

- $H^2(\mathbb{D})$, the $L^2(\mathbb{T})$ closure of the analytic polynomials on $\mathbb{D}$.
- $\mathbb{P} : L^2(\mathbb{T}) \rightarrow H^2(\mathbb{D})$ be the orthogonal projection.
- A Toeplitz operator with symbol $\varphi$ is the following map from $H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$:
  \[ T\varphi(f) \equiv \mathbb{P}(\varphi f). \]
- An important question raised by Sarason is the following:

**Conjecture (Sarason Conjecture)**

The composition of $T\varphi T\psi$ is bounded on $H^2(\mathbb{D})$ if and only if

\[
\sup_{z \in \mathbb{D}} \left( \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - z\bar{\xi}|^2} |\varphi(\xi)|^2 \, dm(\xi) \right) \left( \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - z\bar{\xi}|^2} |\psi(\xi)|^2 \, dm(\xi) \right) < \infty
\]

Unfortunately, this is not true! A counterexample was constructed by Nazarov.
The Sarason Conjecture & Hilbert Transform

Question (Sarason Question (Revised Version))

Obtain necessary and sufficient (testable (?)) conditions so that one can tell if \( T_\varphi T_\psi \) is bounded on \( H^2(\mathbb{D}) \) by evaluating these conditions.

Possible to rephrase this question as one about the two-weight boundedness of the Hilbert transform.

- Let \( M_\phi \) denote multiplication by \( \phi \): \( M_\phi f \equiv \phi f \);
- \( H^2(\|\phi\|^2) \) is the \( L^2(\mathbb{T}) \) closure of \( p\phi \) where \( p \) is an analytic polynomial;

\[
\begin{array}{ccc}
H^2 & \xrightarrow{T_\varphi T_\psi} & H^2 \\
M_\psi \downarrow & \downarrow M_\varphi & \downarrow M_\varphi \\
L^2 \left( \mathbb{T}; |\psi|^{-2} \right) & \xrightarrow{H} & L^2 \left( \mathbb{T}; |\varphi|^2 \right)
\end{array}
\]

Deep work by Nazarov, Treil, Volberg, and then subsequent work by Lacey, Sawyer, Shen, Uriarte-Tuero allow for an answer in terms of the Hilbert transform.
Haar Paraproducts

- $L^2 \equiv L^2(\mathbb{R})$;
- $\mathcal{D}$ is the standard grid of dyadic intervals on $\mathbb{R}$;
- Define the Haar function $h^0_I$ and averaging function $h^1_I$ by

$$h^0_I \equiv h_I \equiv \frac{1}{\sqrt{|I|}} (-1_{I_-} + 1_{I_+}) \quad I \in \mathcal{D}$$

$$h^1_I \equiv \frac{1}{|I|} 1_I \quad I \in \mathcal{D}.$$ 

- $\{h_I\}_{I \in \mathcal{D}}$ is an orthonormal basis of $L^2$.

\[ h^1_{[0,1]}(x) \quad \quad h^0_{[0,1]}(x) \]
Haar Paraproducts from Multiplication Operators

Given a function $b$ and $f$ it is possible to study their pointwise product by expanding in their Haar series:

$$bf = \left( \sum_{I \in \mathcal{D}} \langle b, h_I \rangle_{L^2} h_I \right) \left( \sum_{J \in \mathcal{D}} \langle f, h_J \rangle_{L^2} h_J \right)$$

$$= \sum_{I, J \in \mathcal{D}} \langle b, h_I \rangle_{L^2} \langle f, h_J \rangle_{L^2} h_I h_J$$

$$= \left( \sum_{I=J} + \sum_{I \not\subset J} + \sum_{J \not\subset I} \right) \langle b, h_I \rangle_{L^2} \langle f, h_J \rangle_{L^2} h_I h_J$$

$$= \sum_{I \in \mathcal{D}} \langle b, h_I \rangle_{L^2} \langle f, h_I \rangle_{L^2} h_I^1 + \sum_{I \in \mathcal{D}} \langle b, h_I \rangle_{L^2} \langle f, h_I^1 \rangle_{L^2} h_I$$

$$+ \sum_{I \in \mathcal{D}} \langle b, h_I^1 \rangle_{L^2} \langle f, h_I \rangle_{L^2} h_I.$$
Haar Paraproducts

Definition (Haar Paraproducts)

Given a symbol sequence $b = \{b_I\}_{I \in \mathcal{D}}$ and a pair $(\alpha, \beta) \in \{0,1\}^2$, define the dyadic paraproduct acting on a function $f$ by

$$P_b^{(\alpha,\beta)}f \equiv \sum_{I \in \mathcal{D}} b_I \left\langle f, h_I^{\beta} \right\rangle_{L^2} h_I^{\alpha}.$$ 

The index $(\alpha, \beta)$ is referred to as the type of $P_b^{(\alpha,\beta)}$.

Question (Discrete Sarason Question)

For each choice of pairs $(\alpha, \beta), (\epsilon, \delta) \in \{0,1\}^2$, obtain necessary and sufficient conditions on symbols $b$ and $d$ so that

$$\left\| P_b^{(\alpha,\beta)} \circ P_d^{(\epsilon,\delta)} \right\|_{L^2 \rightarrow L^2} < \infty.$$
Internal Cancellations and Simple Characterizations

When there are internal zeros the behavior of $P_{b}^{(\alpha, 0)} \circ P_{d}^{(0, \beta)}$ reduces to the behavior of $P_{a}^{(\alpha, \beta)}$ for a special symbol $a$. For $f, g \in L^2$, let $f \otimes g : L^2 \to L^2$ be the map given by

$$f \otimes g(h) \equiv f \langle g, h \rangle_{L^2}.$$

Then:

$$P_{b}^{(\alpha, 0)} \circ P_{d}^{(0, \beta)} = \left( \sum_{I \in \mathcal{D}} b_I h_I^{\alpha} \otimes h_I \right) \left( \sum_{J \in \mathcal{D}} d_J h_J \otimes h_J^{\beta} \right)$$

$$= \sum_{I \in \mathcal{D}} b_I d_I h_I^{\alpha} \otimes h_I^{\beta}$$

$$= P_{b \circ d}^{(\alpha, \beta)}.$$

Here $b \circ d$ is the Schur product of the symbols, i.e., $(b \circ d)_I = b_I d_I$. 
Norms and Induced Sequences

For a sequence $a = \{a_I\}_{I \in \mathcal{D}}$ define the following quantities:

\[
\|a\|_{\ell^\infty} \equiv \sup_{I \in \mathcal{D}} |a_I| ;
\]
\[
\|a\|_{CM} \equiv \sqrt{\sup_{I \in \mathcal{D}} \frac{1}{|I|} \sum_{J \subset I} |a_J|^2} .
\]

Associate to $\{a_I\}_{I \in \mathcal{D}}$ two additional sequences indexed by $\mathcal{D}$:

\[
E(a) \equiv \left\{ \frac{1}{|I|} \sum_{J \subset I} a_J \right\}_{I \in \mathcal{D}} ;
\]
\[
\hat{S}(a) \equiv \left\{ \left\langle \sum_{J \in \mathcal{D}} a_J h_J^1, h_I \right\rangle_{L^2} \right\}_{I \in \mathcal{D}} = \left\{ \sum_{J \subsetneq I} a_J \hat{h}_J^1 (I) \right\}_{I \in \mathcal{D}} .
\]
Classical Characterizations

Theorem (Characterizations of Type (0, 0), (0, 1), and (1, 0))

The following characterizations are true:

\[ \left\| P_a^{(0,0)} \right\|_{L^2 \to L^2} = \| a \|_{\ell^\infty}; \]
\[ \left\| P_a^{(0,1)} \right\|_{L^2 \to L^2} = \left\| P_a^{(1,0)} \right\|_{L^2 \to L^2} \approx \| a \|_{CM} . \]

\[ P_a^{(1,1)} = P_a^{(1,0)} \frac{S(a)}{S(a)} + P_a^{(0,1)} \frac{S(a)}{S(a)} + P_a^{(0,0)} E(a) . \]

Theorem (Characterization of Type (1, 1))

The operator norm \( \left\| P_a^{(1,1)} \right\|_{L^2 \to L^2} \) of \( P_a^{(1,1)} \) on \( L^2 \) satisfies

\[ \left\| P_a^{(1,1)} \right\|_{L^2 \to L^2} \approx \left\| \frac{S(a)}{S(a)} \right\|_{CM} + \left\| E(a) \right\|_{\ell^\infty} . \]
Proof of the Easy Characterizations

A simple computation gives

\[ \left\| P_{a}^{(0,0)} f \right\|_{L^2}^2 = \sum_{I, I' \in D} a_I \overline{a_{I'}} \langle f, h_I \rangle_{L^2} \overline{\langle f, h_{I'} \rangle_{L^2}} \langle h_I, h_{I'} \rangle_{L^2} \]

\[ = \sum_{I \in D} |a_I|^2 \left| \langle f, h_I \rangle_{L^2} \right|^2 , \]

\[ \left\| f \right\|_{L^2}^2 = \sum_{I \in D} \left| \langle f, h_I \rangle_{L^2} \right|^2 . \]

Similarly,

\[ \left\| P_{a}^{(0,1)} f \right\|_{L^2}^2 = \sum_{I, I' \in D} a_I \overline{a_{I'}} \langle f, h_I^1 \rangle_{L^2} \overline{\langle f, h_{I'}^1 \rangle_{L^2}} \langle h_I, h_{I'} \rangle_{L^2} \]

\[ = \sum_{I \in D} |a_I|^2 \left| \langle f, h_I^1 \rangle_{L^2} \right|^2 \]
Proof of the Easy Characterizations

Theorem (Carleson Embedding Theorem)

Let \( \{\alpha_I\}_{I \in \mathcal{D}} \) be positive constants. The following two statements are equivalent:

\[
\sum_{I \in \mathcal{D}} \alpha_I \langle f, h_I^1 \rangle_{L^2}^2 \lesssim C \|f\|_{L^2}^2 \quad \forall f \in L^2;
\]

\[
\sup_{I \in \mathcal{D}} \frac{1}{|I|} \sum_{J \subset I} \alpha_J \leq C.
\]

Then the Carleson Embedding Theorem gives

\[
\left\| P_a^{(0,1)} \right\|_{L^2 \to L^2} \lesssim \|a\|_{CM}.
\]

Let \( \hat{I} \) denote the parent of \( I \), and then we have

\[
\left\| P_a^{(0,1)} \right\|_{L^2 \to L^2}^2 \geq \left\| P_a^{(0,1)} h_{\hat{I}} \right\|_{L^2}^2 \gtrsim \frac{1}{|I|} \sum_{J \subset I} |a_J|^2.
\]
Alternate Interpretations: Testing Conditions

It is easy to see for paraproducts of type $(0,0)$ that:

\[
\| P^{(0,0)}_a \|_{L^2 \to L^2} = \| a \|_{\ell^\infty} = \sup_{I \in \mathcal{D}} \| P^{(0,0)}_a h_I \|_{L^2}.
\]

Moreover,

\[
\| P^{(1,0)}_a \|_{L^2 \to L^2} \approx \| P^{(0,1)}_a \|_{L^2 \to L^2} \approx \| a \|_{CM} \approx \sup_{I \in \mathcal{D}} \| P^{(0,1)}_a h_I \|_{L^2}.
\]

These observations suggest seeking a characterization for the other compositions in terms of testing conditions on classes of functions.
Given weights $u$ and $v$ on $\mathbb{R}$ and an operator $T$ a problem one frequently encounters in harmonic analysis is the following:

**Question**

Determine necessary and sufficient conditions on $T$, $u$, and $v$ so that

$$T : L^2 (\mathbb{R}; u) \rightarrow L^2 (\mathbb{R}; v)$$

is bounded.

**Meta-Theorem (Characterization of Boundedness via Testing)**

The operator $T : L^2 (\mathbb{R}; u) \rightarrow L^2 (\mathbb{R}; v)$ is bounded if and only if

$$\| T(u 1_Q) \|_{L^2(v)} \lesssim \| 1_Q \|_{L^2(u)}$$

$$\| T^*(v 1_Q) \|_{L^2(u)} \lesssim \| 1_Q \|_{L^2(v)}.$$
Characterization of Type $(0, 1, 1, 0)$

For a sequence $a$, and interval $I \in \mathcal{D}$ let $Q_I a \equiv \sum_{J \subset I} a_J h_J$.

Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)

The composition $P^{(0,1)}_b \circ P^{(1,0)}_d$ is bounded on $L^2$ if and only if both

$$\left\| Q_I P^{(0,1)}_b P^{(1,0)}_d \left( Q_I \bar{d} \right) \right\|^2_{L^2} \leq C_1^2 \| Q_I d \|^2_{L^2} \quad \forall I \in \mathcal{D};$$

$$\left\| Q_I P^{(0,1)}_d P^{(1,0)}_b \left( Q_I \bar{b} \right) \right\|^2_{L^2} \leq C_2^2 \| Q_I b \|^2_{L^2} \quad \forall I \in \mathcal{D}.$$

Moreover, the norm of $P^{(0,1)}_b \circ P^{(1,0)}_d$ on $L^2$ satisfies

$$\left\| P^{(0,1)}_b \circ P^{(1,0)}_d \right\|_{L^2 \to L^2} \approx C_1 + C_2$$

where $C_1$ and $C_2$ are the best constants appearing above.
Rephrasing the Testing Conditions

We want to rephrase the testing conditions on $Q_I \overline{d}$ and $Q_I \overline{b}$:

$$\left\| Q_I P_b^{(0,1)} P_d^{(1,0)} \left( Q_I \overline{d} \right) \right\|_{L^2}^2 \leq C_1^2 \left\| Q_I d \right\|_{L^2}^2 \quad \forall I \in \mathcal{D};$$

$$\left\| Q_I P_d^{(0,1)} P_b^{(1,0)} \left( Q_I \overline{b} \right) \right\|_{L^2}^2 \leq C_2^2 \left\| Q_I b \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}.$$

It isn’t hard to see that these are equivalent to the following inequalities on the sequences:

$$\sum_{J \subset I} \frac{1}{|J|^2} \left( \sum_{L \subset J} |d_L|^2 \right)^2 \leq C_1^2 \sum_{L \subset I} |d_L|^2 \quad \forall I \in \mathcal{D};$$

$$\sum_{J \subset I} \frac{1}{|J|^2} \left( \sum_{L \subset J} |b_L|^2 \right)^2 \leq C_2^2 \sum_{L \subset I} |b_L|^2 \quad \forall I \in \mathcal{D}.$$
Characterization of Type \((0, 1, 0, 0)\)

**Theorem (E. Sawyer, S. Pott, M. Reguera-Rodriguez, BDW)**

The composition \(P_{b}^{(0,1)} \circ P_{d}^{(0,0)}\) is bounded on \(L^2\) if and only if both

\[
|d_I|^2 \left\| P_{b}^{(0,1)} h_I \right\|_{L^2}^2 \leq C_1^2 \quad \forall I \in \mathcal{D};
\]

\[
\left\| Q_I P_{d}^{(0,0)} P_{b}^{(1,0)} Q_I \overline{b} \right\|_{L^2}^2 \leq C_2^2 \left\| Q_I b \right\|_{L^2}^2 \quad \forall I \in \mathcal{D}.
\]

Moreover, the norm of \(P_{b}^{(0,1)} \circ P_{d}^{(0,0)}\) on \(L^2\) satisfies

\[
\left\| P_{b}^{(0,1)} \circ P_{d}^{(0,0)} \right\|_{L^2 \rightarrow L^2} \approx C_1 + C_2
\]

where \(C_1\) and \(C_2\) are the best constants appearing above.
Motivations

Main Results

Rephrasing Testing Conditions

Again, it is possible to recast the conditions:

\[
\begin{align*}
|d_I|^2 \left\| P_b^{(0,1)} h_I \right\|_{L^2}^2 & \leq C_1^2 \quad \forall I \in D; \\
\left\| QIP_d^{(0,0)} P_b^{(1,0)} Q_I b \right\|_{L^2}^2 & \leq C_2^2 \left\| Q_I b \right\|_{L^2}^2 \quad \forall I \in D
\end{align*}
\]

as expressions depending only on the sequences. In particular, these are equivalent to the following inequalities:

\[
\begin{align*}
\frac{|d_I|^2}{|I|} \sum_{L \not\subseteq I} |b_L|^2 & \leq C_1^2 \quad \forall I \in D; \\
\sum_{J \subset I} \frac{|d_J|^2}{|J|} \left( \sum_{K \subset J_+} |b_K|^2 - \sum_{K \subset J_-} |b_K|^2 \right)^2 & \leq C_2^2 \sum_{L \subset I} |b_L|^2 \quad \forall I \in D.
\end{align*}
\]
Preliminaries

For $I \in \mathcal{D}$ set

$$T(I) \equiv I \times \left[ \frac{|I|}{2}, |I| \right]$$  \hspace{1cm} \text{(Carleson Tile)};

$$Q(I) \equiv I \times [0, |I|] = \bigcup_{J \subset I} T(J)$$  \hspace{1cm} \text{(Carleson Square)}.

- The dyadic lattice $\mathcal{D}$ is in correspondence with the Carleson Tiles.
- Let $\mathcal{H}$ denote the upper half plane $\mathbb{C}_+: \mathcal{H} = \bigcup_{I \in \mathcal{D}} T(I)$.
- For a non-negative function $\sigma$ let $L^2(\mathcal{H}; \sigma)$ denote the functions that are square integrable with respect to $\sigma \, dA$, i.e,

$$\|f\|_{L^2(\mathcal{H}; \sigma)}^2 \equiv \int_{\mathcal{H}} |f(z)|^2 \sigma(z) \, dA(z) < \infty.$$

When $\sigma \equiv 1$, $L^2(\mathcal{H}; 1) \equiv L^2(\mathcal{H})$.

- For $f \in L^2(\mathcal{H})$, let $\widetilde{f} \equiv \frac{f}{\|f\|_{L^2(\mathcal{H})}}$ denote the normalized function.
Functions Constant on Tiles

Let $L^2_c(\mathcal{H}) \subset L^2(\mathcal{H})$ be the subspace of functions which are constant on tiles. Namely, $f : \mathcal{D} \rightarrow \mathbb{C}$

$$f = \sum_{I \in \mathcal{D}} f_I 1_{T(I)}.$$ 

Then

$$L^2_c(\mathcal{H}) \equiv \left\{ f : \mathcal{D} \rightarrow \mathbb{C} : \sum_{I \in \mathcal{D}} |f(I)|^2 |I|^2 < \infty \right\};$$

$$\|f\|_{L^2_c(\mathcal{H})}^2 \equiv \frac{1}{2} \sum_{I \in \mathcal{D}} |f(I)|^2 |I|^2.$$ 

Easy to show:

$$\left\{ \tilde{1}_{T(I)} \right\}_{I \in \mathcal{D}} \text{ is an orthonormal basis of } L^2_c(\mathcal{H});$$

$$\left\{ \tilde{1}_{Q(I)} \right\}_{I \in \mathcal{D}} \text{ is an Riesz basis of } L^2_c(\mathcal{H}).$$
The Gram Matrix of $P_{b}^{(0,1)} \circ P_{d}^{(0,0)}$

Let $\mathcal{G}_{P_{b}^{(0,1)} \circ P_{d}^{(0,0)}} = [G_{I,J}]_{I,J \in \mathcal{D}}$ be the Gram matrix of the operator $P_{b}^{(0,1)} \circ P_{d}^{(0,0)}$ relative to the Haar basis $\{h_{I}\}_{I \in \mathcal{D}}$. A simple computation shows its entries are:

$$G_{I,J} = \left\langle P_{b}^{(0,1)} \circ P_{d}^{(0,0)} h_{J}, h_{I} \right\rangle_{L^{2}} = \left\langle P_{d}^{(0,0)} h_{J}, P_{b}^{(1,0)} h_{I} \right\rangle_{L^{2}}$$

$$= \left\langle d_{J} h_{J}, b_{I} h_{I}^{1} \right\rangle_{L^{2}}$$

$$= \overline{b_{I}} d_{J} \frac{1}{\sqrt{|J|}}$$

$I \subset J_{-}$

$$= \overline{b_{I}} d_{J} \frac{-1}{\sqrt{|J|}}$$

$I \subset J_{+}$

$$0$$

$J \subset I$ or $I \cap J = \emptyset$.

**Idea:** Construct $T_{b,d}^{(0,1,0,0)} : L^{2}_{c}(\mathcal{H}) \rightarrow L^{2}_{c}(\mathcal{H})$ that has the same Gram matrix as $P_{b}^{(0,1)} \circ P_{d}^{(0,0)}$, but with respect to the basis $\left\{ \backslash \left\{ T(I) \right\} \right\}_{I \in \mathcal{D}}$. 
The Operator $T^{(0,1,0,0)}_{b,d}$

Now consider the operator $T^{(0,1,0,0)}_{b,d}$ defined by

$$T^{(0,1,0,0)}_{b,d} \equiv M^{-1}_{\tilde{b}} \left( \sum_{K \in D} \tilde{1}_{Q_{\pm}(K)} \otimes \tilde{1}_{T(K)} \right) M^{\frac{1}{2}}_{d}.$$ 

Here

$$1_{Q_{\pm}(K)} \equiv - \sum_{L \subset K_-} 1_{T(L)} + \sum_{L \subset K_+} 1_{T(L)}.$$

A straightforward computation shows

$$\|1_{Q_{\pm}(K)}\|_{L^2(\mathcal{H})} = \frac{|K|}{2};$$

$$M^{\lambda}_{a} 1_{Q_{\pm}(K)} = - \sum_{L \subset K_-} a_{L} |L|^{\lambda} 1_{T(L)} + \sum_{L \subset K_+} a_{L} |L|^{\lambda} 1_{T(L)}.$$
The Gram Matrix for the Operator $T_{b,d}^{(0,1,0,0)}$

The Gram matrix $\mathcal{G}_{T_{b,d}^{(0,1,0,0)}} = [G_{I,J}]_{I,J \in \mathcal{D}}$ of $T_{b,d}^{(0,1,0,0)}$ relative to the basis $\{\tilde{1}_{T(I)}\}_{I \in \mathcal{D}}$ then has entries given by

$$G_{I,J} = \left\langle T_{b,d}^{(0,1,0,0)} \tilde{1}_{T(J)}, \tilde{1}_{T(I)} \right\rangle_{L^2(\mathcal{H})}$$

$$= \sqrt{2} \begin{cases} 
-\overline{b}_I d_J |J|^{-\frac{1}{2}} & \text{if } I \subset J_- \\
\overline{b}_I d_J |J|^{-\frac{1}{2}} & \text{if } I \subset J_+ \\
0 & \text{if } J \subset I \text{ or } I \cap J = \emptyset.
\end{cases}$$

Thus, up to an absolute constant, $\mathcal{G}_{T_{b,d}^{(0,1,0,0)}} = \mathcal{G}_{P_{b}^{(0,1)} \circ P_{d}^{(0,0)}}$, and so

$$\|P_{b}^{(0,1)} \circ P_{d}^{(0,0)}\|_{L^2 \rightarrow L^2} \approx \|T_{b,d}^{(0,1,0,0)}\|_{L^2(\mathcal{H}) \rightarrow L^2(\mathcal{H})}.$$
Connecting to a Two Weight Inequality

The inequality we wish to characterize is:

\[
\left\| \mathcal{M}_{-1}^{-1} U \mathcal{M}_{\frac{1}{2}}^d f \right\|_{L_c^2(\mathcal{H})} = \left\| T_{b,d}^{(0,1,0,0)} f \right\|_{L_c^2(\mathcal{H})} \lesssim \| f \|_{L_c^2(\mathcal{H})}.
\]

Where the operator \( U \) on \( L^2(\mathcal{H}) \) is defined by

\[
U \equiv \sum_{K \in \mathcal{D}} \tilde{1}_{Q_\pm(K)} \otimes \tilde{1}_{T(K)}.
\]

One sees that the inequality to be characterized is equivalent to:

\[
\| U (\mu g) \|_{L_c^2(\mathcal{H};\nu)} \lesssim \| g \|_{L_c^2(\mathcal{H};\mu)},
\]

where the weights \( \mu \) and \( \nu \) are given by

\[
\nu \equiv \sum_{I \in \mathcal{D}} |b_I|^2 |I|^{-2} 1_{T(I)}
\]

and

\[
\mu \equiv \sum_{I \in \mathcal{D}} |d_I|^{-2} |I|^{-1} 1_{T(I)}.
\]
Theorem (S. Pott, E. Sawyer, M. Reguera-Rodriguez, BDW)

Let

\[ U \equiv \sum_{K \in \mathcal{D}} \tilde{1}_{Q_\pm(K)} \otimes \tilde{1}_{T(K)} \]

and suppose that \( \mu \) and \( \nu \) are positive measures on \( \mathcal{H} \) that are constant on tiles, i.e., \( \mu \equiv \sum_{I \in \mathcal{D}} \mu_I \mathbf{1}_{T(I)} \), \( \nu \equiv \sum_{I \in \mathcal{D}} \nu_I \mathbf{1}_{T(I)} \). Then

\[ U(\cdot \mu) : L^{2}_c(\mathcal{H}; \mu) \rightarrow L^{2}_c(\mathcal{H}; \nu) \]

if and only if both

\[ \| U(\mu \mathbf{1}_{T(I)}) \|_{L^2_c(\mathcal{H}; \nu)} \leq C_1 \| \mathbf{1}_{T(I)} \|_{L^2_c(\mathcal{H}; \mu)} = \sqrt{\mu(T(I))}, \]

\[ \| \mathbf{1}_{Q(I)} U^* (\nu \mathbf{1}_{Q(I)}) \|_{L^2_c(\mathcal{H}; \mu)} \leq C_2 \| \mathbf{1}_{Q(I)} \|_{L^2_c(\mathcal{H}; \nu)} = \sqrt{\nu(Q(I))}, \]

hold for all \( I \in \mathcal{D} \). Moreover, \[ \| U \|_{L^2_c(\mathcal{H}; \mu) \rightarrow L^2_c(\mathcal{H}; \nu)} \approx C_1 + C_2. \]
An Application: Linear Bound for Hilbert Transform

- For a weight $w$, i.e., a positive locally integrable function on $\mathbb{R}$, let $L^2(w) \equiv L^2(\mathbb{R}; w)$.
- A weight belongs to $A_2$ if: $[w]_{A_2} \equiv \sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < +\infty$.
- The Hilbert transform is the operator: $H(f)(x) \equiv \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{y-x} \, dy$.

**Theorem (Petermichl)**

Let $w \in A_2$. Then $\|H\|_{L^2(w) \to L^2(w)} \lesssim [w]_{A_2}$, and the linear growth is optimal.

- $\|T\|_{L^2(w) \to L^2(w)} = \|M_{w^{1/2}} TM_{w^{-1/2}}\|_{L^2 \to L^2}$;
- $H$ is the average of dyadic shifts $\mathbb{III}$;
- $M_{w^{1/2}} \mathbb{III} M_{w^{-1/2}}$ can be written as a sum of nine compositions of paraproducts; Some of which are amenable to the Theorems above.
- However, each term can be shown to have norm no worse than $[w]_{A_2}$. 

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An Open Question

Unfortunately, the methods described do not appear to work to handle type $(0, 1, 0, 1)$ compositions. However, the following question is of interest:

**Question**

For each $I \in D$ determine function $F_I, B_I \in L^2$ of norm 1 such that $P_b^{(0,1)} \circ P_d^{(0,1)}$ is bounded on $L^2$ if and only if

$$
\left\| P_b^{(0,1)} \circ P_d^{(0,1)} F_I \right\|_{L^2} \leq C_1 \quad \forall I \in D;
$$

$$
\left\| P_d^{(1,0)} \circ P_b^{(1,0)} B_I \right\|_{L^2} \leq C_2 \quad \forall I \in D.
$$

Moreover, we will have

$$
\left\| P_b^{(0,1)} \circ P_d^{(0,1)} \right\|_{L^2 \rightarrow L^2} \approx C_1 + C_2.
$$
(Modified from the Original Dr. Fun Comic)

Thanks to Jie and Kehe for Organizing the Meeting!
Thank You!