The Essential Norm of Operators on the Bergman Space

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Thanks to the Organizers

(Modified from the Original Dr. Fun Comic)

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Weighted Bergman Spaces on $\mathbb{B}_n$

- Let $\mathbb{B}_n := \{ z \in \mathbb{C}^n : |z| < 1 \}$.
- For $\alpha > -1$, we let

$$dv_\alpha(z) := c_\alpha (1 - |z|^2)^\alpha \, dv(z), \quad \text{with} \quad c_\alpha := \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)}.$$  

The choice of $c_\alpha$ gives that $v_\alpha(\mathbb{B}_n) = 1$.

- For $1 < p < \infty$ the space $A^p_\alpha$ is the collection of holomorphic functions on $\mathbb{B}_n$ such that

$$\|f\|_{A^p_\alpha} := \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) < \infty.$$  

- For $\lambda \in \mathbb{B}_n$ let

$$k^{(p,\alpha)}_\lambda(z) = \frac{(1-|\lambda|^2)^{n+1+\alpha}}{(1-\lambda z)^{n+1+\alpha}}.$$  

A computation shows: $\|k^{(p,\alpha)}_\lambda\|_{A^p_\alpha} \approx 1$.  

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Toeplitz Operators and the Toeplitz Algebra

- The projection of $L^2_\alpha$ onto $A^2_\alpha$ is given by the integral operator
  
  $P_\alpha(f)(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1 - zw)^{n+1+\alpha}} d\nu_\alpha(w).$

- This operator is bounded from $L^p_\alpha$ to $A^p_\alpha$ when $1 < p < \infty$ and $-1 < \alpha$.

- Let $M_a$ denote the operator of multiplication by the function $a$, $M_a(f) := af$. The Toeplitz operator with symbol $a \in L^\infty$ is the operator given by
  
  $T_a := P_\alpha M_a.$

- It is immediate to see that $\|T_a\|_{L^p(A^p_\alpha)} \lesssim \|a\|_{L^\infty}$.

- More generally, for a measure $\mu$ we will define the operator
  
  $T_\mu f(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \overline{w}z)^{n+1+\alpha}} d\mu(w),$

  which will define an analytic function for all $f \in H^\infty$. 
Motivations for the Problem

Toeplitz Operators and the Toeplitz Algebra

• For symbols in $L^\infty$ we let $\mathcal{T}_{p,\alpha}$ be the $C^*$ subalgebra of $\mathcal{L}(A^p_{\alpha})$ generated by $T_a$.

• An important class of operators in $\mathcal{T}_{p,\alpha}$ are those that are finite sums of finite products of Toeplitz operators. Namely, for symbols $a_{jk} \in L^\infty$ with $1 \leq j \leq J$ and $1 \leq k \leq K$ we will need to study the operators:

$$
\sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}
$$

• Additionally,

$$
\mathcal{T}_{p,\alpha} = \left\{ \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}} : a_{jk} \in L^\infty \quad 1 \leq j \leq J \quad 1 \leq k \leq K \right\} \mathcal{L}(A^p_{\alpha})
$$
Geometry of the Ball

For $z \in \mathbb{B}_n$, $\varphi_z$ will denote the automorphism of $\mathbb{B}_n$ such that $\varphi_z(0) = z$. The pseudohyperbolic and hyperbolic metrics are defined by

$$\rho(z, w) := |\varphi_z(w)| \quad \text{and} \quad \beta(z, w) := \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

The hyperbolic disc centered at $z$ of radius $r$ is denoted by

$$D(z, r) := \{ w \in \mathbb{B}_n : \beta(z, w) \leq r \} = \{ w \in \mathbb{B}_n : \rho(z, w) \leq \tanh r \}.$$

Lemma (Lattices on $\mathbb{B}_n$)

Given $r > 0$, there is a family of Borel sets $D_m \subset \mathbb{B}_n$ and points $\{w_m\}_{m=1}^\infty$ such that

(i) $D\left(w_m, \frac{r}{4}\right) \subset D_m \subset D\left(w_m, r\right)$ for all $m$;
(ii) $D_k \cap D_l = \emptyset$ if $k \neq l$;
(iii) $\bigcup_m D_m = \mathbb{B}_n$. 
Geometry of the Ball

Dyadic Tree on $\mathbb{D}$
Geometry of the Ball

Note that for these sets: If \( w \in D_m \) then \( (1 - |w|^2) \approx (1 - |w_m|^2) \) and \( |1 - \bar{z}w| \approx |1 - \bar{z}w_m| \) uniformly in \( z \in B_n \).

Lemma (Whitney Decompositions)

*There is a positive integer \( N = N(n) \) such that for any \( \sigma > 0 \) there is a covering of \( B_n \) by Borel sets \( \{B_j\} \) that satisfy:*

1. \( B_j \cap B_k = \emptyset \) if \( j \neq k \);
2. Every point of \( B_n \) is contained in at most \( N \) sets \( \Omega_\sigma(B_j) = \{z : \beta(z, B_j) \leq \sigma\} \);
3. There is a constant \( C(\sigma) > 0 \) such that \( \text{diam}_\beta B_j \leq C(\sigma) \) for all \( j \).

Idea of Proof: Via the Whitney Decomposition of the unit ball \( B_n \), partition into dyadic “cubes.” This then gives (i) immediately. The remaining points are then well known geometric facts.
Geometry of the Ball

Whitney Decomposition of $\mathbb{D}$
(Taken from *Classical and Modern Fourier Analysis* by Grafakos)
Motivations for the Problem

Geometry of the Ball

Let $\sigma > 0$ and $k$ a non-negative integer. Let $\{B_j\}$ be the covering of the ball from the previous Lemma with $(k + 1)\sigma$ instead of $\sigma$.

For $0 \leq i \leq k$ and $j \geq 1$ write

$$F_{0,j} = B_j \quad \text{and} \quad F_{i+1,j} = \{z : \beta(z, F_{i,j}) \leq \sigma\}.$$

Corollary

Let $\sigma > 0$ and $k$ be a non-negative integer. For each $0 \leq i \leq k$ the family of sets $\mathcal{F}_i = \{F_{i,j} : j \geq 1\}$ forms a covering of $\mathbb{B}_n$ such that

(i) $F_{0,j_1} \cap F_{0,j_2} = \emptyset$ if $j_1 \neq j_2$;
(ii) $F_{0,j} \subset F_{1,j} \subset \cdots \subset F_{k+1,j}$ for all $j$;
(iii) $\beta(F_{i,j}, F_{i+1,j}^c) \geq \sigma$ for all $0 \leq i \leq k$ and $j \geq 1$;
(iv) Every point of $\mathbb{B}_n$ belongs to no more than $N$ elements of $\mathcal{F}_i$;
(v) $\text{diam}_\beta F_{i,j} \leq C(k, \sigma)$ for all $i, j$. 
Carleson Measures for $A^p_\alpha$

A measure $\mu$ on $\mathbb{B}_n$ is a Carleson measure for $A^p_\alpha$ if

$$
\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \lesssim \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) \quad \forall f \in A^p_\alpha.
$$

Lemma (Characterizations of $A^p_\alpha$ Carleson Measures)

Suppose that $1 < p < \infty$ and $\alpha > -1$. Let $\mu$ be a measure on $\mathbb{B}_n$ and $r > 0$. The following quantities are equivalent, with constants that depend on $n, \alpha$ and $r$:

1. $\|\mu\|_{CM} := \sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\bar{z}w|^{2(n+1+\alpha)}} \, d\mu(w)$;

2. $\|\nu_p\| := \inf \left\{ C : \left( \int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) \right)^{\frac{1}{p}} \right\}$;

3. $\|\mu\|_{Geo} := \sup_{z \in \mathbb{B}_n} \frac{\mu(D(z,r))}{(1-|z|^2)^{n+1+\alpha}}$;

4. $\|T_\mu\|_{L(A^p_\alpha)}$. 
Motivations for the Problem

The Berezin Transform

For $S \in \mathcal{L}(A^p_{\alpha})$, we define the Berezin transform by

$$B(S)(z) := \left\langle S k_z^{(p,\alpha)}, k_z^{(q,\alpha)} \right\rangle_{A^2_{\alpha}}.$$

- $B : \mathcal{L}(A^p_{\alpha}) \to L^\infty(\mathbb{B}_n)$:

$$|B(S)(z)| \leq \|S\|_{\mathcal{L}(A^p_{\alpha})} \|k^{(p,\alpha)}_\lambda\|_{A^p_{\alpha}} \|k^{(q,\alpha)}_\lambda\|_{A^q_{\alpha}} \approx \|S\|_{\mathcal{L}(A^p_{\alpha})}.$$

- If $S$ is compact, then $B(S)(z) \to 0$ as $|z| \to 1$:

$$|B(S)(z)| \leq \|Sk^{(p,\alpha)}_\lambda\|_{A^p_{\alpha}} \|k^{(q,\alpha)}_\lambda\|_{A^q_{\alpha}} \approx \|Sk^{(p,\alpha)}_\lambda\|_{A^p_{\alpha}}.$$

However, $k^{(p,\alpha)}_\lambda \to 0$ as $|z| \to 1$ and so $\|Sk^{(p,\alpha)}_\lambda\|_{A^p_{\alpha}} \to 0$. 
The Berezin Transform

- The Berezin transform is one-to-one: Enough to show that \( B(S)(z) = 0 \Rightarrow S = 0 \).

Set \( F(z, w) = \langle S k_z^{(p,\alpha)}, k_w^{(q,\alpha)} \rangle_{A_\alpha^2} \).

Then \( F(z, z) = 0 \) and \( F \) is analytic in the first variable and anti-analytic in the second variable.

This implies that \( F \) is identically zero.

So we have that \( S k_z^{(p,\alpha)} = 0 \) for all \( z \in \mathbb{B}_n \), or \( S = 0 \).

- \( B(S) \) is Lipschitz continuous with respect to the hyperbolic metric

\[
|B(S)(z_1) - B(S)(z_2)| \leq \sqrt{2} \|S\|_{\mathcal{L}(A_{\alpha}^p)} \beta(z_1, z_2)
\]

- Range of \( B \) is not closed: \( B^{-1} : B(\mathcal{L}(A_{\alpha}^p)) \rightarrow \mathcal{L}(A_{\alpha}^p) \) is not bounded.
Motivations for the Problem

Related Results


Suppose that $a_{jk} \in L^\infty(\mathbb{D})$ with $1 \leq j \leq J$ and $1 \leq k \leq K$. Let $S = \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}$. The following are equivalent:

(a) The operator $S$ is compact on $A^2(\mathbb{D})$;

(b) $B(S)(z) \to 0$ as $|z| \to 1$;

(c) $\|Skz\|_{A^2_\alpha} \to 0$ as $|z| \to 1$.

- The interesting implication is $(b) \Rightarrow (a)$;
- The same proof works in the case of the unit ball, but was done by Raimondo.

Theorem (Engliš, Ark. Mat. 30 (1992))

Let $1 < p < \infty$ and $\alpha > -1$. If $S$ is a compact operator on $A^p_\alpha$, then $S \in T_{p,\alpha}$.
Main Question of Interest

From the previous Theorem and simple functional analysis we have that if $S$ is compact on $A^p_\alpha$ then

$$S \in \mathcal{T}_{p,\alpha} \quad \text{and} \quad B(S)(z) \rightarrow 0 \text{ as } |z| \rightarrow 1.$$  

**Yes!**

- Shown to be true by Suárez for $A^p$ when $1 < p < \infty$ and $\alpha = 0$.
- Extended to $\alpha > -1$ by Suárez, Mitkovski and BDW using some maximal ideal theory and appropriate modifications of the original proof.
- Alternate proof obtained by Mitkovski and BDW, but removing the maximal ideal theory.
Main Results

Characterizations of Compactness and Essential Norm

**Theorem (D. Suárez, M. Mitkovski and BDW)**

Let $1 < p < \infty$ and $\alpha > -1$ and $S \in \mathcal{L}(A_{\alpha}^p)$. Then $S$ is compact if and only if $S \in \mathcal{T}_{p,\alpha}$ and $\lim_{|z| \to 1} \mathcal{B}(S)(z) = 0$.

We can actually obtain much more precise information about the essential norm of an operator. For $S \in \mathcal{L}(A_{\alpha}^p)$ recall that

$$\|S\|_e = \inf \left\{ \|S - Q\|_{\mathcal{L}(A_{\alpha}^p)} : Q \text{ is compact} \right\}.$$ 

We need to define other measures of the “size” of an operator $S \in \mathcal{L}(A_{\alpha}^p)$:

$$b_S := \sup_{r>0} \limsup_{|z| \to 1} \left\| M_{1D(z,r)} S \right\|_{\mathcal{L}(A_{\alpha}^p,L_{\alpha}^p)}$$

$$c_S := \lim_{r \to 1} \left\| M_{1r\mathbb{B}_n^c} S \right\|_{\mathcal{L}(A_{\alpha}^p,L_{\alpha}^p)}.$$ 

In the last definition, we have that $r\mathbb{B}_n^c := \mathbb{B}_n \setminus r\mathbb{B}_n$. 

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Characterizations of Compactness and Essential Norm

Let $r > 0$ and let $\{w_m\}$ and $D_m$ be the sets that form the lattice in $\mathbb{B}_n$. Define the measure

$$\mu_r = \sum_m v_\alpha(D_m) \delta_{w_m} \approx \sum_m (1 - |w_m|^2)^{\alpha+n+1} \delta_{w_m}.$$ 

It is well known that $\mu_r$ is a $A^p_\alpha$ Carleson measure, so $T_{\mu_r} : A^p_\alpha \to A^p_\alpha$ is bounded.

**Lemma**

$T_{\mu_r} \to \text{Id}$ on $\mathcal{L}(A^p_\alpha)$ when $r \to 0$.

Let $r > 0$ be chosen so that $\|T_{\mu_r} - \text{Id}\|_{\mathcal{L}(A^p_\alpha)} < \frac{1}{4}$, and $\mu := \mu_r$. Then set

$$a_S(\rho) := \limsup_{|z| \to 1} \sup_{D(z,\rho)} \left\{ \|Sf\|_{A^p_\alpha} : f \in T_{\mu 1_{D(z,\rho)}}(A^p_\alpha), \|f\|_{A^p_\alpha} \leq 1 \right\}$$

and define

$$a_S := \lim_{\rho \to 1} a_S(\rho).$$
Characterizations of Compactness and Essential Norm

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $1 < p < \infty$ and $\alpha > -1$ and let $S \in T_{p,\alpha}$. Then there exists constants depending only on $n, p,$ and $\alpha$ such that:

$$a_S \approx b_S \approx c_S \approx \|S\|_e.$$

For the automorphism $\varphi_z$ such that $\varphi_z(0) = z$ define the map

$$U_z^{(p,\alpha)} f(w) := f(\varphi_z(w)) \frac{(1 - |z|^2)^{n+1+\alpha}}{p} \frac{2(n+1+\alpha)}{(1 - wz)^{n+1+\alpha}}.$$

A standard change of variable argument and computation gives that

$$\left\| U_z^{(p,\alpha)} f \right\|_{A^p_\alpha} = \|f\|_{A^p_\alpha} \quad \forall f \in A^p_\alpha.$$
Characterizations of Compactness and Essential Norm

For $z \in \mathbb{B}_n$ and $S \in \mathcal{L}(A^p_\alpha)$ we then define the map

$$ S_z := U_z^{(p,\alpha)} S(U_z^{(q,\alpha)})^* . $$

One should think of the map $S_z$ in the following way. This is an operator on $A^p_\alpha$ and so it first acts as “translation” in $\mathbb{B}_n$, then the action of $S$, then “translation” back.

**Theorem (D. Suárez, M. Mitkovski and BDW)**

*Let $\alpha > -1$ and $1 < p < \infty$ and $S \in \mathcal{T}_{p,\alpha}$. Then*

$$ \|S\|_e \approx \sup_{\|f\|_{A^p_\alpha} = 1} \limsup_{|z| \to 1} \|S_z f\|_{A^p_\alpha} . $$
Lemma (D. Suárez, M. Mitkovski and BDW)

Let $S \in T_{p,\alpha}$, $\mu$ a Carleson measure and $\epsilon > 0$. Then there are Borel sets $F_j \subset G_j \subset B_n$ such that

(i) $B_n = \bigcup F_j$;
(ii) $F_j \cap F_k = \emptyset$ if $j \neq k$;
(iii) each point of $B_n$ lies in no more than $N(n)$ of the sets $G_j$;
(iv) $\text{diam}_B G_j \leq d(p, S, \epsilon)$

and

$$\left\| ST_\mu - \sum_{j=1}^{\infty} M_{1_{F_j}} ST_{1_{G_j}} \mu \right\|_{\mathcal{L}(A^p_{\alpha}, L^p_{\alpha})} < \epsilon.$$
A Uniform Algebra and its Maximal Ideal Space

- Let $A$ denote the bounded functions that are uniformly continuous from the metric space $(\mathbb{B}_n, \rho)$ into the metric space $(\mathbb{C}, |\cdot|)$.
- Associate to $A$ its maximal ideal space $M_A$ which is the set of all non-zero multiplicative linear functionals from $A$ to $\mathbb{C}$.
- Since $A$ is a $C^*$ algebra we have that $\mathbb{B}_n$ is dense in $M_A$.
- The Toeplitz operators associated to symbols in $A$ are useful to study the Toeplitz algebra $T_{p,\alpha}$.

Theorem (D. Suárez, M. Mitkovski and BDW)

The Toeplitz algebra $T_{p,\alpha}$ is equal to the closed algebra generated by $\{T_a : a \in A\}$.
A Uniform Algebra and its Maximal Ideal Space

- For an element $x \in M_A \setminus B_n$ choose a net $z_\omega \to x$.
- Form $S_{z_\omega}$ and look at the limit operator obtained when $z_\omega \to x$, denote it by $S_x$.

**Lemma (D. Suárez, M. Mitkovski and BDW)**

\[
\text{Let } S \in \mathcal{L}(A^p_\alpha). \text{ Then } B(S)(z) \to 0 \text{ as } |z| \to 1 \text{ if and only if } S_x = 0 \text{ for all } x \in M_A \setminus B_n.
\]

We can extend this to compute the essential norm of an operator $S$ in terms of $S_x$ where $x \in M_A \setminus B_n$.

**Theorem (D. Suárez, M. Mitkovski and BDW)**

\[
\text{Let } S \in \mathcal{T}_{p,\alpha}. \text{ Then there exists a constant } C(p, \alpha, n) \text{ such that }
\] 
\[
\sup_{x \in M_A \setminus B_n} \|S_x\|_{\mathcal{L}(A^p_\alpha)} \approx \|S\|_e.
\]
The Hilbert Space Case

- More precise information can be obtained in terms of the essential norm and the essential spectral radius.
- Let $\mathcal{K}$ denote the ideal of compact operators on $A^2_\alpha$.
- Recall that the Calkin algebra is given by $\mathcal{L}(A^2_\alpha)/\mathcal{K}$.
- The spectrum of $S$ will be denoted by $\sigma(S)$, and the spectral radius will be denoted by
  \[ r(S) = \sup \{|\lambda| : \lambda \in \sigma(S)\} . \]
- Define the essential spectrum as the spectrum of $S + \mathcal{K}$ in the Calkin algebra, and the essential spectral radius as
  \[ r_e(S) = \sup \{|\lambda| : \lambda \in \sigma_e(S)\} . \]
The Hilbert Space Case

Theorem (D. Suárez, M. Mitkovski and BDW)

For $S \in \mathcal{T}_{2,\alpha}$ we have

$$\|S\|_e = \sup_{x \in M_A \setminus B_n} \|S_x\|_{L(A^2_{\alpha})}$$

and

$$\sup_{x \in M_A \setminus B_n} r(S_x) \leq \lim_{k \to \infty} \left( \sup_{x \in M_A \setminus B_n} \left\| S_x^k \right\|_{L(A^2_{\alpha})}^{1/k} \right) = r_e(S)$$

with equality when $S$ is essentially normal.

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $\alpha > -1$ and $S \in \mathcal{T}_{2,\alpha}$. Then

$$\|S\|_e = \sup_{\|f\|_{A^2_{\alpha}} = 1} \limsup_{|z| \to 1} \|S_zf\|_{A^2_{\alpha}}.$$
The Hilbert Space Case

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $S \in \mathcal{T}_{2,\alpha}$. The following are equivalent:

1. $\lambda \notin \sigma_e(S)$;
2. $\lambda \notin \bigcup_{x \in M_A \setminus B_n} \sigma(S_x)$ and $\sup_{x \in M_A \setminus B_n} \| (S_x - \lambda I)^{-1} \|_{\mathcal{L}(A_{2\alpha}^2)} < \infty$;
3. There is a number $t > 0$ depending only on $\lambda$ such that
   $$\| (S_x - \lambda I)f \|_{A_{2\alpha}^2} \geq t \| f \|_{A_{2\alpha}^2}$$
   and
   $$\| (S_x^* - \lambda I)f \|_{A_{2\alpha}^2} \geq t \| f \|_{A_{2\alpha}^2}$$
   for all $f \in A_{2\alpha}^2$ and $x \in M_A \setminus B_n$. 
Removing the Maximal Ideal Space

Lemma (M. Mitkovski and BDW)

Let $T$ be a finite sum of finite products of Toeplitz operators on $A^2_{\alpha}$. For every $\epsilon > 0$ there exists $r > 0$ such that for the covering $\mathcal{F}_r = \{ F_j \}$ associated to $r$

$$\left\| \sum_j M_{1_{F_j}} TP_{\alpha} M_{1_{G_j^c}} \right\|_{\mathcal{L}(A^2_{\alpha})} < \epsilon.$$ 

Proposition (M. Mitkovski and BDW)

Let $T$ be a finite sum of finite products of Toeplitz operators on $A^2_{\alpha}$. Then

$$\| T \|_e \approx \sup_{\| f \|_{A^2_{\alpha}} = 1} \limsup_{|z| \to 1} \| Tzf \|_{A^2_{\alpha}}.$$
Other Directions

The Fock space $\mathcal{F}$ is the collection of holomorphic functions $f$ on $\mathbb{C}^n$ such that

$$\|f\|_{\mathcal{F}}^2 := \int_{\mathbb{C}^n} |f(z)|^2 e^{-\pi|z|^2} dv(z) < \infty.$$ 

This is a reproducing kernel Hilbert space with $k_\lambda(z) = e^{\pi z \overline{\lambda}}$ as kernel. Similar results are true for the Fock Space.

**Theorem (W. Bauer and J. Isralowitz)**

Let $S \in \mathcal{L}(\mathcal{F})$. Then $S$ is compact if and only if $S \in \mathcal{T}_2$ and $\lim_{|z| \to \infty} B(S)(z) = 0$.

**Corollary (W. Bauer and J. Isralowitz)**

Let $S \in \mathcal{T}_2$, then

$$\|S\|_e \approx \sup_{\|f\|_{\mathcal{F}} = 1} \limsup_{|z| \to \infty} \|Sz f\|_{\mathcal{F}}.$$
Open Questions

If $S \in \mathcal{L}(A^2_\alpha)$ then $B(S) \in L^\infty$. Similarly, if $S \in \mathcal{K}(A^2_\alpha)$ then $B(S) \to 0$ as $|z| \to 1$. And, even better, $S \in \mathcal{K}(A^2_\alpha)$ if and only if $S \in T_{2,\alpha}$ and $B(S) \to 0$ as $|z| \to 1$.

Question

Can we characterize the Schatten class operators on $A^2_\alpha$ as those that belong to the Toeplitz algebra $T_{2,\alpha}$ and an integrability condition on the Berezin transform $B(S)(z)$?

One can show that if $S \in S_p$ then

$$\|B(S)\|_{L^p(B_n;\lambda_n)} := \left( \int_{B_n} |B(S)(z)|^p \, d\lambda_n(z) \right)^{1/p} \lesssim \|S\|_{S_p}.$$ 

Proposition (M. Mitkovski and BDW)

Let $1 < p < \infty$ and $\alpha > -1$. If $S \in S_p$, then $B(S) \in L^p(B_n;\lambda_n)$ and $S \in T_{2,\alpha}$. 

B. D. Wick (Georgia Tech) Essential Norm on Bergman Spaces GPOTS 2012 29 / 31
Open Questions

Let $\Omega$ be a bounded symmetric domain in $\mathbb{C}^n$. These are Hermitian symmetric spaces with a complete Riemannian metric given by the Bergman metric. For each $a \in \Omega$ there is a biholomorphic automorphism $\varphi_a$ that interchanges $0$ and $a$.

Let $A^2(\Omega)$ denote the Bergman space of analytic functions on $\Omega$ that are square integrable with respect to volume measure. This space has a reproducing kernel $K_a$, and relates to the automorphisms by

$$K_w(z) = K_{\varphi(w)}(\varphi(z))J_c \varphi(w)J_c \varphi(w)$$

Conjecture (M. Mitkovski and BDW)

Let $S \in T_2$, then

$$\|S\|_e \approx \sup_{\|f\|_{A^2(\Omega)}=1} \limsup_{z \to \partial \Omega} \|S_z f\|_{A^2(\Omega)}.$$
Thank You!