Homework 8

1. If $b_n > 1$ for all $n$, the prove that $\prod_n b_n$ converges if and only if $\sum_n \log b_n < \infty$.

2. Let $\{a_j\} \subset \mathbb{C} \setminus \{-1\}$. Prove that if

$$\lim_{n \to \infty} \prod_{j=1}^n (1 + a_j)$$
exists and is nonzero, then $a_j \to 0$.

3. If $|z| < R$, then prove that

$$\prod_{n=0}^{\infty} \left( \frac{R^{2n} + z^{2n}}{R^{2n}} \right) = \frac{R}{R - z}.$$ 

4. Calculate explicitly

$$\prod_{n=2}^{\infty} \left( 1 - \frac{1}{n^2} \right).$$

5. Suppose that $\sum |\alpha_n - \beta_n| < \infty$. Determine the largest set of $z$ such that:

$$\prod_{n=1}^{\infty} \frac{z - \alpha_n}{z - \beta_n}$$
converges normally.

6. Let $z_0 \in D_r(0)$ be fixed. Let $f$ be holomorphic on a neighborhood of $\overline{D_r(0)}$. Let $a_1, \ldots, a_n$ be the zeros of $f$ in $D_r(0)$ and assume that no zero lies on $\partial D_r(0)$. Let $\phi(z) = \frac{z^2 + r^2}{r + z^2}$. Apply Jensen’s formula to $f \circ \phi$ and do a change of variable in the integral to obtain the formula:

$$\log |f(z_0)| + \sum_{k=1}^{n} \log \left| \frac{r^2 - a_k z_0}{r(z_0 - a_k)} \right| = \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left( \frac{r e^{i\theta} + z_0}{r e^{i\theta} - z_0} \right) \log |f(r e^{i\theta})| \, d\theta.$$ 

7. Prove that if $f$ is holomorphic in the unit disc, bounded and not identically zero, and if $z_1, z_2, \ldots, z_n, \ldots$ are its zeros $|z_k| < 1$, then:

$$\sum_n (1 - |z_n|) < \infty.$$