Math 5022 - Complex Analysis 2
Homework 1

1. Let U be an open subset of $\mathbb{C}$. Show that $u(z) := -\log \operatorname{dist}(z, \partial U)$ is subharmonic on U.

2. Let u be a subharmonic function on $\mathbb{D}$ such that $u < 0$. Prove that for each $\xi \in \mathbb{T}$ that:

$$\limsup_{r \to 1-} \frac{u(r\xi)}{1-r} < 0.$$

Hint: Apply the maximum principle to $u(z) + c \log|z|$ on the set $\{1/2 < |z| < 1\}$ for suitable constant c.

3. Let $v(z)$ be a positive subharmonic function on the unit disc and define:

$$m(r) = \frac{1}{2\pi} \int v(re^{i\theta}) d\theta.$$

Then $m(r)$ is an increasing convex function of $\log r$. That is, if $\log r = t \log r_1 + (1-t) \log r_2$, $0 < t < 1$, then:

$$m(r) \leq tm(r_1) + (1-t)m(r_2).$$

4. Define the hyperbolic length of an arc γ in $\mathbb{D}$ to be:

$$\int_\gamma \frac{2\,|dz|}{1-|z|^2}.$$

Define the Poincaré metric $\psi(z_1, z_2)$ as the infiumum of the hyperpolic lengths of arcs joining z_1 to z_2.

- Show that ψ is confromally invariant.
- Compute the distance between $z_1 = 0$ and $z_2 = r > 0$ explicitly.
- Show that $\psi(z, w) = \log \frac{1+\rho(z,w)}{1-\rho(z,w)}$ where $\rho(z, w) = \left|\frac{z-w}{1-\bar{z}w}\right|$.

5. Let $K(z_0, r)$ denote the non-Euclidean disc

$$K(z_0, r) = \{z : \rho(z, z_0) < r\}$$

where ρ is the pseudohyperbolic distance. Prove that $K(z_0, r)$ is also a Euclidean disc $D_R(c)$ with $c = c(r, z_0)$ and $R = R(r, z_0)$. Determine explicit formulae for the center c and the radius r.