Math 5022 - Complex Analysis 2
Homework 2

1. Let $0 < p < \infty$ and $f \in H^p(\mathbb{D})$ with $f \not\equiv 0$. Let $\{z_n\}$ be the zeros of f and B the Blaschke product with zeros $\{z_n\}$. Then $g = \frac{f}{B}$ is in $H^p(\mathbb{D})$ and

$$\|f\|_{H^p} = \|g\|_{H^p}.$$

2. If $f \in H^p(\mathbb{D})$ for $1 \leq p \leq \infty$ show that for $z \in \mathbb{D}$ we have:

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\xi)}{\xi - z} d\xi.$$

3. For $1 \leq p \leq \infty$, if $f \in H^p(\mathbb{D})$ then

(i) $|f(z)| \leq \frac{\|f\|_{H^p}}{(1-|z|^2)^{\frac{1}{p}}}$;

(ii) $\left|f^{(n)}(z)\right| \leq C_{n,p} \frac{\|f\|_{H^p}}{(1-|z|^2)^{n+\frac{1}{p}}}$;

(iii) Show that up to constants, are sharp for every p by considering the function $f(z) = \left(\frac{1-|a|^2}{(1-\overline{a}z)}\right)^{\frac{1}{p}}$ with $a \in \mathbb{D}$.

4. Plemjl Jump Formula. Let $f \in L^1(\mathbb{D})$ and $z \in \mathbb{D}$. Then:

(i)

$$\frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - \frac{1}{\overline{z}}} d\xi = \int f(e^{i\theta}) P_z(\theta) d\theta.$$

(ii) Conclude that for almost every point in $\mathbb{T}$

$$\lim_{r \to 1} \left(\frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - re^{i\theta}} d\xi - \frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - \frac{1}{r} e^{i\theta}} d\xi \right) = f(e^{i\theta}).$$

5. Let $f \in H^2(\mathbb{D})$ prove that:

(i)

$$\lim_{r \to 1} \int_{\mathbb{T}} \left|f(re^{i\theta})\right|^2 d\theta = \int_{\mathbb{T}} \left|f(e^{i\theta})\right|^2 d\theta.$$

(ii)

$$\lim_{r \to 1} \int_{\mathbb{T}} \left|f(re^{i\theta}) - f(e^{i\theta})\right|^2 d\theta = 0.$$

Hint: Let $f(z) = \sum_n a_n z^n$ and then use that:

$$\int_{\mathbb{T}} \left|f(re^{i\theta}) - f(e^{i\theta})\right|^2 d\theta \leq \liminf_{\rho \to 1} \int_{\mathbb{T}} \left|f(re^{i\theta}) - f(\rho e^{i\theta})\right|^2 d\theta$$

6. Suppose that $f \in H^p$. Then prove that $f = gh$ with $g, h \in H^{2p}$ and $\|g\|_{2p} = \|h\|_{2p} = \|f\|_p^{1/2}$.

7. For $\alpha > 1$ and $\xi \in \mathbb{T}$ let $\Gamma_\alpha(\xi) = \{z \in \mathbb{D} : |\xi - z| < \alpha(1 - |z|)\}$. Let $f \in L^1(\mathbb{T})$ and let $u(z)$ be the Poisson extension of f. Then prove that:

$$\sup_{z \in \Gamma(\xi)} |u(z)| \leq C_\alpha M(f)(\xi)$$

where $Mf(\xi) = \sup_{I \ni \xi} \frac{1}{|\xi|} \int_I \left| f(e^{i\theta}) \right| d\theta$. Prove that if $f \in L^1(\mathbb{T})$ then $\lim_{z \in \Gamma_\alpha(\xi), z \to \xi} u(z) = f(\xi)$ for almost every $\xi \in \mathbb{T}$.