Math 5022 - Complex Analysis 2
Homework 4

1. Use Runge's Theorem to prove the following statement: If $\Omega \subset \mathbb{C}$ is an open set and if $f : \Omega \to \mathbb{C}$ is holomorphic, then there exists a sequence $\{r_j\}$ of rational functions with poles in $\mathbb{C} \cup \{\infty\} \setminus \Omega$ such that $r_j \to f$ normally on Ω.

2. Define $g(z) = \frac{z}{e^z - 1}$ and write $g(z) = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k$. Calculate that

1. $\zeta(1-2m) = (-1)^m \frac{B_{2m}}{2m}$;
2. $\zeta(2m) = \frac{2^{2m-1}\pi^{2m} B_{2m}}{(2m)!}$

for $m = 1, 2, 3, \ldots$.

3. Let $\{a_n\}$ be a sequence of complex numbers. The function $u(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$ is called a *Dirichlet series*. Let σ_0 be a real number. Prove that if the Dirichlet series converges for some value of z with $\operatorname{Re} z = \sigma_0$, then it converges for all z with $\operatorname{Re} z > \sigma_0$, uniformly on every compact subset of this region.

4. A sequence of numbers $\{a_n\}$ is *strongly multiplicative* if $a_1 = 1$ and $a_{mn} = a_m a_n$ for all m and n. Show that:

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p \in \mathcal{P}} (1 - a_p p^{-s})^{-1}$$

provided the infinite series/product converges.

5. Prove the following converse to Runge's Theorem. If K is a compact set whose complement is not connected, then there exists a function f holomorphic in a neighborhood of K which can not be approximated uniformly by a polynomial on K. Hint: Pick z_0 belonging to a bounded component of K^c and let $f(z) = (z - z_0)^{-1}$. Proceed by contradiction to arrive at a polynomial such that $|(z - z_0)p(z) - 1| < 1$. Use Maximum Modulus to deduce this holds for all z in the bounded component of containing z_0.

6. Define the Riemann surface R of $\log z$ in terms of explicit coordinate patches. Define explicitly the function on R determined by $\log z$. Show that it is a $1-1$ analytic map of R onto the complex plane.

7. Show that the analytic maps from a Riemann surface R to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ are the meromorphic functions on R and the constant function ∞.