Math 5022 - Complex Analysis 2
Homework 5

1. Recall that $\tilde{h}(z) = \sup\{u(z) : \tilde{F}_h\}$ with $\mathcal{F}_h$ is the family of subharmonic functions $u(z)$ on D such that $\limsup_{D \ni z \to \zeta} u(z) \leq h(\zeta)$, for $\zeta \in D$. Let D be the annulus $\{a < |z| < b\}$. Find $\tilde{h}(z)$ for the function $h(\zeta) = \alpha$ for $|\zeta| = a$ and $h(\zeta) = \beta$ for $|\zeta| = b$.

2. Prove the following Strict Maximum Principle: If u is a subharmonic function on a Riemann surface R, and u attains its maximum at some point of R, then u is constant on R.

3. Let $w = w(z)$ be analytic on a domain D in the complex plane. Show that

$$\frac{\partial^2}{\partial z \partial \overline{z}} = \left|\frac{dw}{dz}\right|^2 \frac{\partial^2}{\partial w \partial \overline{w}}.$$

Deduce that a smooth function $h(w)$ is harmonic on $w(D)$ if and only if $h(w(z))$ is harmonic on D.

4. Show that if Green's function exists for S, and R is a subsurface of S, then Green's function exists for R, and $g_R \leq g_S$.

5. Suppose Green's function $g(p,q)$ exists for R. Let $z(p)$ be a coordinate map at q with $z(q) = 0$. Show that if $u(p)$ is a subharmonic function on $R \setminus \{q\}$ such that $u(p) = 0$ off some compact subset of R, and $u(p) + \log|z(p)|$ is bounded above near q, then $u(p) \leq g(p,q)$.

6. For τ in the upper half plane, denote L_τ by the lattice $\mathbb{Z} + \tau\mathbb{Z}$ generated by 1 and τ, and denote the Riemann surface $\mathbb{C}/L_\tau$ by T_τ.

(a) Show that the Riemann surface $T = \mathbb{C}/L$, where L is the lattice generated by two complex numbers ω_1 and ω_2 that do not lie on the same line through the origin, is conformally equivalent to the Reimann surface T_τ for some τ in the upper half plane. Hint: Take $\tau = \pm\omega_1/\omega_2$, with the sign chosen so that $\operatorname{Im}\tau > 0$.

(b) Show that T_τ is conformally equivalent to $T_{\tau'}$ if and only if there is a fractional linear transformation of the form $f(z) = \dfrac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{Z}$ satisfy $ad - bc = 1$ such that $f(\tau) = \tau'$.