Math 5022 - Complex Analysis 2
Homework 6

1. Let $\{f_j\}$ be a sequence of C^1 functions on a domain Ω such that $\overline{\partial} f_j = 0$ for each j. Suppose that the sequence converges uniformly on compact sets to a limit function f. Prove that the limit function is C^1 and satisfies $\overline{\partial} f = 0$ on Ω. Hint: Apply Bochner-Martinelli on a smoothly bounded subdomain Ω' that is compactly contained in Ω to f_j. Differentiate under the integral sign to prove that $f_j \to f$ in the C^1 topology on any subdomain Ω'' of Ω'.

2. Prove that a C^2 function f is pluriharmonic if and only if $\partial_j \overline{\partial}_k f = 0$ for all $j, k = 1, \ldots, n$. Show that this is in turn true if and only if $\partial\overline{\partial} f = 0$.

3. If $\Omega \subset \mathbb{C}^n$ is holomorphic, then $\log |f|$ and $|f|^p$ is plurisubharmoinc.

4. Prove that $f \in C^2(\Omega)$ is plurisubharmonic if and only if $\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} w_j \overline{w}_k \geq 0$ for every $z \in \Omega$ and for every $w \in \mathbb{C}^n$.

5. It M is a positive semi-definite $m \times m$ matrix and if $F = (f_1, \ldots, f_m$ is a m tuple of holomorphic functions, then prove that $\langle MF, F \rangle_{\mathbb{C}^n}$ is plurisubharmoinc.

6. Prove the Cauchy estimates for a function that is holomorphic on the polydisc $\prod_{j=1}^{n} D_{r_j}(p_j)$.