MATH 6337: HOMEWORK 12 SOLUTIONS

9.1. Use Minkowski's integral inequality to prove (9.1): if $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}^n)$, and $g \in L^1(\mathbb{R}^n)$, then $f * g \in L^p(\mathbb{R}^n)$ and $||f * g||_p \leq ||f||_p \, ||g||_1$.

Solution. The cases of $p = 1, \infty$ can be dealt with as in the text. For $1 < p < \infty$, Minkowski's integral inequality states that

$$\left|\left| \int f(x,y)\, dx \right|\right|_p \leq \int ||f(x,y)||_p \, dx,$$

where the norms are taken with respect to the y-variable. Letting $F(x,y) = f(y-x)g(x)$, we have

$$f * g(y) = \int F(x,y)\, dx,$$

so

$$||f * g||_p = \left|\left| \int F(x,y)\, dx \right|\right|_p \leq \int ||f(y-x)g(x)||_p \, dx = \int |g(x)| \, ||f(y-x)||_p \, dx = ||f||_p \, ||g||_1 \, .$$

The penultimate equality follows because g is a function of x, but the norm is taken with respect to y. The last equality follows because integrals over $\mathbb{R}^n$ are invariant under translations. $\square$

9.3. Show that if $f \in L^p(\mathbb{R}^n)$ and $K \in L^q(\mathbb{R}^n)$, $1 \le p \le \infty$ and $1/p + 1/q = 1$, then $f * K$ is bounded and continuous in $\mathbb{R}^n$.

Solution. By Young's convolution theorem (see Problem 9.2), $||f * K||_\infty \le ||f||_p \, ||K||_q < +\infty$, so $f * K$ is bounded by $c = ||f * K||_\infty$ a.e. in $\mathbb{R}^n$. (Showing that $f * K$ is continuous will prove that $f * K$ is bounded everywhere by c.)

If $1 < p \le +\infty$, then by Hölder's inequality

$$|f * K(x+h) - f * K(x)| \le \int |f(t)| \, |K(x+h-t) - K(x-t)| \, dt \le ||f||_p \left|\left|\widetilde{K}(t-h) - \widetilde{K}(t)\right|\right|_q,$$

where $\widetilde{K}(t) = K(x-t)$. Since $\widetilde{K} \in L^q$, we have by continuity in L^q that $\left|\left|\widetilde{K}(t-h) - \widetilde{K}(t)\right|\right|_q \to 0$ as $|h| \to 0$. Since $||f||_p < +\infty$, we've proven continuity of $f * K$.

If $p = 1$ (so that $q = +\infty$), then switch the roles of K and f:

$$|f * K(x+h) - f * K(x)| \le \int |K(t)| \, |f(x+h-t) - f(x-t)| \, dt \le ||K||_\infty \left|\left|\widetilde{f}(t-h) - \widetilde{f}(t)\right|\right|_1 \to 0.$$

$\square$

9.5. Let G, G_1 be bounded open subsets of $\mathbb{R}^n$ such that $\overline{G_1} \subset G$. Construct a function $h \in C_0^\infty$ such that $h = 1$ in G_1 and $h = 0$ outside G. [Hint: Choose an open G_2 such that $\overline{G_1} \subset G_2$ and $\overline{G_2} \subset G$. Let $h = \mathbb{1}_{G_2} * K$ for a $K \in C^\infty$ with suitably small support and $\int K = 1$.

Solution. Let

$$K(x) = Ce^{-1/(1-|x|^2)}\mathbb{1}_{B_1(0)},$$

where C is chosen so that $\int K = 1$. Choose $\varepsilon < \min(\operatorname{dist}(G_1, \partial G_2), \operatorname{dist}(G_2, \partial G))$, and define $K_\varepsilon = \frac{1}{\varepsilon^n} K(x/\varepsilon)$, so that $\int K_\varepsilon = 1$. Also, $\operatorname{supp} K_\varepsilon = \overline{B_\varepsilon(0)}$ and $K_\varepsilon \in C_0^\infty$.

Choose G_2 as described in the hint, and define $h(x) = \mathbb{1}_{G_2} * K_\varepsilon(x)$. Since $\mathbb{1}_{G_2} \in L^1$ (as G_2 is bounded) and $K \in C_0^\infty$, we have $h \in C_0^\infty$. Also, for $x \in G_1$, we have

$$h(x) = \int_{\mathbb{R}^n} \mathbb{1}_{G_2}(x-t)K(t)\,dt = \int_{B_\varepsilon(0)} \mathbb{1}_{G_2}(x-t)K(t)\,dt = \int_{B_\varepsilon(0)} K(t)\,dt = 1$$

since $\mathbb{1}_(G_2)(x-t) = 1$ for such points and $|t| < \varepsilon$; if $x \notin G$, then we have

$$h(x) = \int_{\mathbb{R}^n} \mathbb{1}_{G_2}(x-t)K(t)\,dt = \int_{B_\varepsilon(0)} \mathbb{1}_{G_2}(x-t)K(t)\,dt = 0$$

since $\mathbb{1}_{G_2}(x-t) = 0$ for such points and $|t| < \varepsilon$. $\square$

9.9. The maximal function is defined as $f^*(x) = \sup |Q|^{-1} \int_Q |f|$, where the supremum is taken over cubes Q with center x. Let $f^{**}(x)$ be defined similarly, but with the supremum taken over all Q *containing* x. Thus, $f^*(x) \leq f^{**}(x)$. Show that there is a positive constant c depending only on the dimension such that $f^{**}(x) \leq cf^*(x)$.

Solution. Write $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. For each $i = 1, ..., n$ let $\delta_i = \max(x_i - a_i, b_i - x_i)$, and let $\delta = \max_i \delta_i$. Let Q' be the cube centered at x with each side length 2δ: then $Q \subset Q'$, so

$$\int_Q |f| \leq \int_{Q'} |f|,$$

and $|Q'| \leq 2^n |Q|$, so

$$\frac{1}{|Q|} \int_Q |f| \leq \frac{2^n}{|Q'|} \int_{Q'} |f|.$$

So, for every Q containing x, there exists Q' centered at x such that the above holds. Thus, $f^{**}(x) \leq 2^n f^*(x)$. $\square$