MATH 6337: Homework 2 Solutions

3.3. Construct a two-dimensional Cantor set in the unit square $[0,1] \times [0,1]$ as follows: subdivide the square into nine equal parts and keep on the four closed corner squares, removing the remaining region (which forms a cross). The repeat this process in a suitably scaled version for the remaining squares, *ad infinitum*. Show that the resulting set is perfect, has plane measure zero, and equals $\mathcal{C} \times \mathcal{C}$.

Solution. Let S_0 be the unit square, let S_k be the set of points remaining after k iterations of the process described above, and let $\mathcal{S} = \bigcap_{k=0}^{\infty} S_k$. [Remark: $S_k \searrow \mathcal{S}$.] Observe that $\mathcal{S}$ is a closed set, since $\mathcal{S}^c$ is a union of open intervals and is therefore open.[1]

Note that if Q is a corner point of one of the disjoint closed squares comprising some S_k, then $Q \in \mathcal{S}$, since Q is a corner point for some square in S_j for all $j \geq k$. Moreover, if $P \in \mathcal{S}$, then for every $k \geq 0$ there exists a (unique) closed square T_k containing P. Let Q_k be a corner point of T_k which is not P:[2] then P is a limit point of the sequence Q_k, since $|P - Q_k| < \operatorname{diam}(S_k) = \sqrt{2} \cdot 3^{-k}$. Since each $Q_k \in \mathcal{S}$ (as we've already established that corner points are in $\mathcal{S}$), P is a limit point of $\mathcal{S}$; hence $\mathcal{S}$ is a perfect set.

$\mathcal{S}$ is measurable because it is the intersection of a countable collection of measurable sets. Also, S_k consists of 4^k disjoint closed squares, each of area 9^{-k}, so $|S_k| = (\frac{4}{9})^k$. As $k \to \infty$, $|S_k| \to 0$, and since $\mathcal{S} \subseteq S_k$ for all k, by the monotonicity of measure we see that $|\mathcal{S}| = 0$.

We prove by induction that $S_k = C_k \times C_k$ for all $k \geq 0$. Clearly $S_0 = C_0 \times C_0$. Now suppose $S_k = C_k \times C_k$, and we'll show that $S_{k+1} = C_{k+1} \times C_{k+1}$. Let $P = (x, y) \in S_k$, and let T be the closed square of S_k in which P lies. If $P \in S_k$, then P is in one of the four closed subsquares of T. By construction, this means that x is in either the first or last third of the interval $\pi_x(T)$ (the projection of T onto the x-axis), which by the inductive hypothesis lies in C_k. Thus, $x \in C_{k+1}$, and a similar argument shows that $y \in C_{k+1}$. On the other hand, if $P \notin S_{k+1}$, then P lies in the open cross removed from T, so either x or y lies in the middle third of $\pi_x(T)$ or $\pi_y(T)$, respectively, so $P \notin C_{k+1} \times C_{k+1}$.

Since $S_k = C_k \times C_k$ for all k, we have

$$\mathcal{S} = \bigcap_k S_k = \bigcap_k (C_k \times C_k) = \left(\bigcap_k C_k\right) \times \left(\bigcap_k C_k\right) = \mathcal{C} \times \mathcal{C}.^3$$

□

[1] You need to argue that $\mathcal{S}$ is closed, since a set must be closed in order to be perfect.

[2] It is not sufficient to pick the sequence of point converging to P to be just the sequence of bottom-left (for example) corner points; if P *is* the bottom-left corner point, then P is not a limit point of this sequence.

[3] The elements of $\mathcal{C}$ are *not* just the endpoints of the removed intervals (in this case $\mathcal{C}$ would be countable!). For example, $1/4 \in \mathcal{C}$ but is not the endpoint of an interval in any of the C_k. [To see that $1/4 \in \mathcal{C}$, observe that $\sum_{k=1}^{\infty} 2 \cdot 9^{-k} = 1/4$, so $1/4$ has the ternary representation 0.020202... and is therefore in $\mathcal{C}$.]

3.4. Construct a subset of $[0,1]$ in the same manner as the Cantor set by removing from each remaining interval a subinterval of relative length θ, $0 < \theta < 1$. Show that the resulting set is perfect and has measure zero.

Solution. Let $C_0^\theta = [0,1]$, let C_k^θ be the set of points remaining after k iterations of the process described above, and let $\mathcal{C}^\theta = \bigcap_{k=0}^\infty C_k^\theta$. [Remark: $C_k^\theta \searrow \mathcal{C}^\theta$.] Observe that $\mathcal{C}^\theta$ is a closed set, since $(\mathcal{C}^\theta)^c$ is a union of open intervals and is therefore open.[4]

Note that if t is an endpoint of one of the disjoint closed intervals comprising some C_k^θ, then $t \in \mathcal{C}^\theta$, since t is an endpoint of some closed interval in C_j^θ for all $j \geq k$. Moreover, if $x \in \mathcal{C}^\theta$, then for every $k \geq 0$ there exists a (unique) closed interval J_k containing x. Let t_k be an endpoint of J_k which is not x:[5] then x is a limit point of the sequence t_k, since $|x - t_k| < (1-\theta)^k/2$. Since each $t_k \in \mathcal{C}^\theta$ (as we've already established that endpoints are in $\mathcal{C}^\theta$), x is a limit point of $\mathcal{C}^\theta$; hence $\mathcal{C}^\theta$ is a perfect set.

$\mathcal{C}^\theta$ is measurable because it is the intersection of a countable collection of measurable sets. Also, C_k^θ consists of 2^k disjoint closed intervals, each of length $(\frac{1-\theta}{2})^k$, so $\left|C_k^\theta\right| = (1-\theta)^k$. As $k \to \infty$, $\left|C_k^\theta\right| \to 0$, and since $\mathcal{C}^\theta \subseteq C_k^\theta$ for all k, by the monotonicity of measure we see that $\left|\mathcal{C}^\theta\right| = 0$. $\square$

[4] Again, you need to argue that $\mathcal{C}^\theta$ is closed in order to show that it is perfect.

[5] Again, your argument must handle the case that x is an endpoint of J_k.

3.5. Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the k^{th} stage, each interval removed has length $\delta 3^{-k}$, $0<\delta<1$. Show that the resulting set is perfect, has measure $1-\delta$, and contains no intervals.

Solution. Let $C_0^\delta = [0,1]$, let C_k^δ be the set of points remaining after k iterations of the process described above, and let $\mathcal{C}^\delta = \bigcap_{k=0}^\infty C_k^\delta$. [Remark: $C_k^\delta \searrow \mathcal{C}^\delta$.]

Since each closed interval in C_n^δ has the same size and since there are 2^n such intervals, the length of such an interval is bounded above by $\frac{1-\left|(C_n^\delta)^c\right|}{2^n} < 2^{-n}$. Since $\mathcal{C}^\delta$ is closed, since endpoints of intervals in C_n^δ are contained in $\mathcal{C}^\delta$, and since the distance between $x \in \mathcal{C}^\delta$ and an endpoint of an interval in C_n^δ is less than 2^{-n}, we conclude that $\mathcal{C}^\delta$ is a perfect set.

$\mathcal{C}^\delta$ is measurable because it is the intersection of a countable collection of measurable sets. To find its measure, we will first compute $\left|(\mathcal{C}^\delta)^c\right|$, where we take the complement with respect to $[0,1]$. $(\mathcal{C}^\delta)^c$ is the union of the increasing sequence of sets $(C_k^\delta)^c$. Going from $(C_k^\delta)^c$ to $(C_{k+1}^\delta)^c$, we add 2^k open intervals of length $\delta 3^{-(k+1)}$, so

$$\left|(C_n^\delta)^c\right| = \frac{\delta}{3}\sum_{k=0}^{n-1}\left(\frac{2}{3}\right)^k.$$

Then

$$\lim_{n\to\infty}\left|(C_n^\delta)^c\right| = \frac{\delta}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \frac{\delta}{3}\cdot\frac{1}{1-(2/3)} = \delta.$$

By the continuity of measure from below, $\left|(\mathcal{C}^\delta)^c\right| = \lim_{n\to\infty}\left|(C_n^\delta)^c\right| = \delta$. Thus, $\left|C_n^\delta\right| = 1-\delta$.

If an interval (a,b) were contained in $\mathcal{C}^\delta$, then it would have to be contained in C_k^δ for all k. But the connected components of C_k^δ have length less than 2^{-k}, so (a,b) cannot be contained in C_k^δ if k is large enough. Thus, $\mathcal{C}^\delta$ contains no intervals. $\square$

3.6. Construct a Cantor-type subset of $[0,1]$ by removing from each interval remaining at the k^{th} stage a subinterval of relative length θ_k, $0 < \theta_k < 1$. Show that the remainder has measure zero if and only if $\sum \theta_k = +\infty$. (Use the fact that for $a_k > 0$, $\prod_{k=1}^{\infty} a_k$ converges in the sense that $\lim_{N\to\infty} \prod_{k=1}^{N} a_k$ exists and is not zero if and only if $\sum_{k=1}^{\infty} \log(a_k)$ converges.)

Solution. Let $\Theta_0 = [0,1]$, let Θ_k be the set of points remaining after k iterations of the process described above, and let $\Theta = \bigcap_{k=0}^{\infty} \Theta_k$. [Remark: $\Theta_k \searrow \Theta$.]

Θ is measurable because it is the intersection of a countable collection of measurable sets. Also, Θ_n consists of 2^n disjoint closed intervals, each of length $\prod_{k=1}^{n} \frac{1-\theta_k}{2^k}$, so

$$|\Theta_n| = \prod_{k=1}^{n} (1-\theta_k).$$

Since each factor of the product is positive but strictly less than one, $|\Theta_n|$ is a strictly decreasing sequence bounded below by 0, so it must converge to some real number, and by the continuity of measure from above (and since all the measures involved are finite) we are assured that $|\Theta| = \lim_{n\to\infty} |\Theta_n|$.

Now, either $\theta_k \not\to 0$, in which case $\theta_k \geq c > 0$ for some fixed c, or else $\theta_k \to 0$. In the former case, $\sum \theta_k = +\infty$, and we can compute that, as $n \to \infty$,

$$|\Theta_n| = \prod_{k=1}^{n} (1-\theta_k) \leq \prod_{k=1}^{n} (1-c) = (1-c)^n \to 0.$$

In the latter case, we can use l'Hôpital's rule to show that

$$\lim_{k\to\infty} \frac{\theta_k}{\log(1-\theta_k)} = \lim_{x\to 0^+} \frac{x}{\log(1-x)} = \lim_{x\to 0^+} \frac{1}{-1/(1-x)} = -1.$$

Since $\lim_{k\to\infty} \frac{\theta_k}{\log(1-\theta_k)}$ exists and is positive, the limit comparison test implies that $\sum \theta_k$ converges if and only if $\sum \log(1-\theta_k)$ converges. We finish the proof by invoking the fact that $\lim_{n\to\infty} |\Theta_n|$ converges to a nonzero number if and only if $\sum_{k=1}^{\infty} \log(1-\theta_k)$ converges. $\square$

3.9. If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ (and so also $\liminf E_k$) has measure zero.

Solution. Let $E = \limsup E_k$ and $F_j = \bigcup_{k=j}^{\infty} E_k$; then F_j is a decreasing sequence of sets whose limit is $\limsup E_k$. Then $E \subset \bigcap_{j=n}^{\infty} F_j \subseteq F_n$ for all n, so $|E|_e \leq |F_n|_e$ by monotonicity of outer measure. Moreover,

$$|F_n|_e \leq \sum_{k=n}^{\infty} |E_k|_e$$

for all n. Since $\sum |E_k|_e$ converges, for every $\varepsilon > 0$ there exists N such that $\sum_{k=n}^{\infty} |E_k|_e < \varepsilon$ for all $n \geq N$.[6] Thus, for every $\varepsilon > 0$, $|E|_e \leq \varepsilon$, so $|E|_e = 0$.[7] Thus, $E = \limsup E_k$ is measurable and has measure 0. Since $\liminf E_k \subseteq \limsup E_k$, $\liminf E_k$ also has measure zero. □

[6] It is not sufficient to only use the fact that $|E|_e \to 0$. Consider the sequence of (measurable!) sets $\{E_k\}$ given by $E_0 = [0,1]$, $E_1 = [0,1/2]$, $E_2 = [1/2,1]$, $E_3 = [0,1/3]$, $E_4 = [1/3,2/3]$, $E_5 = [2/3,1]$, $E_6 = [0,1/4]$, etc. Then $|E_k| \to 0$ but $\limsup E_k = [0,1]$ with measure 1.

[7] Unlike measure, outer measure is not "continuous from above," i.e., it is not necessarily true that if $E_k \searrow E$ and $|E_k|_e < +\infty$ for all k, then $\lim_{k\to\infty} |E_k|_e \to |E|_e$. (See Exercise 3.21.)