MATH 6337: Homework 6 Solutions

5.1. If f is a simple measurable function (not necessarily positive) taking values a_j on E_j, $j = 1, 2, ..., N$, show that

$$\int_E f = \sum_{j=1}^{N} a_j \left| E_j \right|.$$

[Hint: Use (5.24).*]

Solution. Using the hint, we have

$$\int_E f = \sum_{j=1}^{N} \int_{E_j} f = \sum_{j=1}^{N} \int_{E_j} a_j = \sum_{j=1}^{N} a_j \left| E_j \right|.$$

□

5.3. Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \leq f$ a.e. on E, show that

$$\int_E f_k \to \int_E f.$$

Solution. By Fatou's lemma,

$$\int_E f = \int_E \liminf f_k \leq \liminf \int_E f_k,$$

and since $f_k \leq f$ a.e., we have

$$\int_E f_k \leq \int_E f$$

for all k, so

$$\limsup \int_E f_k \leq \int_E f.$$

Thus

$$\liminf \int_E f_k = \limsup \int_E f_k = \int_E f.$$

□

*If $\int_E f$ exists and $E = \bigcup_k E_k$ is the countable union of disjoint measurable sets E_k, then $\int_E f = \sum_k \int_{E_k} f$.

5.4. If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, ...$ and that

$$\int_0^1 x^k f(x)\, dx \to 0.$$

Solution. On $(0,1)$, $x^k \leq 1$, so

$$\int_0^1 \left|x^k f(x)\right| \, dx = \int_0^1 x^k \left|f(x)\right| \, dx \leq \int_0^1 \left|f(x)\right| \, dx < +\infty.$$

Let $f_k(x) = x^k f(x)$ and let $g(x) = |f(x)|$ (so g is integrable on $(0,1)$). Since $x^k \to 0$ everywhere on $(0,1)$, we also have $f_k(x) \to 0$ a.e. in $(0,1)$. Since $|f_k(x)| = x^k |f(x)| \leq g(x)$ for all k, we apply the dominated convergence theorem:

$$\int_0^1 x^k f(x) = \int_0^1 f_k(x) \to \int_0^1 \lim_{k\to\infty} f_k(x) = 0.$$

□

5.5. Use Egorov's theorem to prove the bounded convergence theorem.

Solution. Suppose $\{f_k\}$ is a sequence of measurable functions such that $f_k \to f$ a.e. in E, where $|E| < +\infty$, and so that $|f_k(x)| \leq M$ for all $x \in E$. Let $\varepsilon > 0$. By Egorov's theorem there exists a closed set $F \subseteq E$ such that $|E \setminus F| < \varepsilon$ and on which $f_k \to f$ uniformly. In particular, there exists K such that for $k > K$, $|f_k - f| < \varepsilon$ on F. Thus

$$\left|\int_E f_k - f\right| \leq \int_{E\setminus F} |f_k - f| + \int_F |f_k - f| \leq \int_{E\setminus F} 2M + \int_F \varepsilon \leq$$
$$2M\,|E \setminus F| + \varepsilon\,|E| < (2M + |E|)\varepsilon.$$

Since $|E| < +\infty$, we conclude that for every $\varepsilon > 0$ there exists $K > 0$ such that $\left|\int_E f_k - f\right| < c\varepsilon$ for some constant c, so $\int_E f_k \to \int_E f$ as $k \to \infty$. □

5.9. If $p > 0$ and

$$\int_E |f - f_k|^p \to 0$$

as $k \to \infty$, show that $f_k \xrightarrow{m} f$ on E (and thus there is a subsequence $f_{k_j} \to f$ a.e. in E).[†]

Solution. We prove the contrapositive. Suppose that there is $\varepsilon > 0$ such that $|X| \geq \varepsilon$ for all k, where $X = \{|f - f_k| > \varepsilon\}$.[‡] Then

$$\int_E |f - f_k|^p \geq \int_X |f - f_k|^p \geq \int_X \varepsilon^p \geq \varepsilon^{p+1} > 0$$

for all ε.[§] □

5.10. If $p > 0$,

$$\int_E |f - f_k|^p \to 0, \quad \text{and} \quad \int_E |f_k|^p \leq M \text{ for all } k,$$

show that

$$\int_E |f|^p \leq M.$$[¶]

Solution. Using the previous problem, we know that $f_k \xrightarrow{m} f$; in particular, there exists a subsequence $f_{k_j} \to f$ a.e. in E. Thus, by Fatou's lemma,

$$\int_E |f|^p = \int_E \liminf \left|f_{k_j}\right|^p \leq \liminf \int_E \left|f_{k_j}\right|^p \leq M.$$[‖]

□

[†]It might make more sense to understand this statement in probabilistic terms. Suppose $f_k, k \geq 1$, and f are random variables and that the expected value of $|f_k - f|^p$ is zero. Then, for every $\varepsilon > 0$, the probability that $|f_k - f| > \varepsilon$ approaches zero as k becomes large. In general, there is a translation between measure-theoretic concepts and probabilistic concepts: measure is the same as probability, the Lebesgue integral is the same as the expected value, convergence in measure is the same as convergence in probability (which I just described), etc.

[‡]See Problem 4.16 from last week's homework.

[§]It is not true that since $\int_E |f - f_k|^p \to 0$, then $|f - f_k|^p$ a.e. Let $E = [0, 1]$ and let f_k be defined by the following pattern: $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/3]}$, $f_5 = \chi_{[1/3,2/3]}$, etc. Then $\int_E |f - f_k|^p \to 0$ but $|f(x) - f_k(x)|^p$ doesn't converge for any x. This is an example of the principle that convergence in p^{th} mean doesn't imply convergence almost everywhere; you've already seen examples of functions which converge a.e. but don't converge in p^{th} mean.

[¶]In probabilistic terms, this means, "If $f_k \to f$ in p^{th} mean and the p^{th} moments of the f_k are bounded by M, then so is the p^{th} moment of f."

[‖]Many of you tried to use a triangle inequality involving $\int |f|^p$. The correct form of the triangle inequality is $\left(\int |f+g|^p\right)^{1/p} \leq \left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}$ for $p \geq 1$; the inequality becomes $\left(\int |f+g|^p\right)^{1/p} \leq K_p \left[\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right]$ for a constant K_p when $0 < p < 1$.

5.11. For which $p > 0$ is $1/x \in L^p(0,1)$? $L^p(1,\infty)$? $L^p(0,\infty)$?

Solution. $1/x \in L^p(0,1)$ if and only if

$$\int_0^1 \frac{1}{x^p}\,dx < +\infty.$$

By standard calculus arguments, this is true precisely when $p < 1$. Similarly, $1/x \in L^p(1,\infty)$ if and only if

$$\int_1^\infty \frac{1}{x^p}\,dx < +\infty,$$

which is true precisely when $p > 1$.

$1/x \in L^p(0,\infty)$ if and only if

$$\int_0^\infty \frac{1}{x^p}\,dx = \int_0^1 \frac{1}{x^p}\,dx + \int_1^\infty \frac{1}{x^p}\,dx < +\infty,$$

which is true if and only if the integrals over $(0,1)$ and $(1,\infty)$ both converge. However, from the above we see that no choice of p causes both of these integrals to converge, so $1/x \notin L^p(0,\infty)$ for any $p > 0$. $\square$