

Homework 6

1. Let $S, T \in B(X)$ show that for any $\lambda \in \rho(T) \cap \rho(S)$ that

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(S - T)R_\lambda(S).$$

Solution: Note that

$$T_\lambda R_\lambda(T) = I \quad R_\lambda(S)S_\lambda = I.$$

Then we have

$$\begin{aligned} R_\lambda(S)(T - S)R_\lambda(T) &= R_\lambda(S)(T_\lambda - S_\lambda)R_\lambda(T) \\ &= (R_\lambda(S)T_\lambda - I)R_\lambda(T) \\ &= R_\lambda(S) - R_\lambda(T) \end{aligned}$$

2. Let X be a complex Banach space, with $T \in B(X)$ and p a polynomial. Show that the equation

$$p(T)x = y$$

has a unique solution x for every $y \in X$ if and only if $p(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$.

Solution: Note that if $p(\lambda) \neq 0$ for $\lambda \in \sigma(T)$ then the function $g = \frac{1}{p}$ is holomorphic on a set that contains $\sigma(T)$. But we have that $pg = 1$, and by the Spectral Theorem $p(T)g(T) = I$, and so $p(T)$ is invertible, and thus has a unique solution x for all $y \in X$.

Conversely, if there is a unique solution then $p(T)$ is invertible. Suppose that $p(\lambda) = 0$ for some $\lambda \in \sigma(T)$, then we have $p(z) = (z - \lambda)h(z)$, which gives

$$(T - \lambda I)h(T) = p(T) = h(T)(T - \lambda I).$$

But since $T - \lambda I$ would not be invertible, we would have that $p(T)$ is not invertible, and this is a contradiction and so $p(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$.

3. Let T be a bounded operator on a Banach space. Suppose that $T^2 = T$, but $T \neq 0$ and $T \neq I$. Compute $\sigma(T)$.

Solution: Note that $T^2 - T = 0$, and so $p(T) = 0$ where $p(z) = z^2 - z$. By the Spectral Theorem we have

$$\{0\} = \sigma(p(T)) = p(\sigma(T)).$$

So points $\lambda \in \sigma(T)$ must satisfy the equation

$$p(\lambda) = 0 = \lambda^2 - \lambda = \lambda(\lambda - 1).$$

Thus $\sigma(T) = \{0, 1\}$.

4. Let $X \neq \{0\}$ be a complex Banach space. A bounded linear operator T is *nilpotent* if there is a positive integer m such that $T^m = 0$. Suppose that $T : X \rightarrow X$, compute $\sigma(T)$.

Solution: By the Spectral Theorem we have that

$$\sigma(T^m) = (\sigma(T))^m$$

But since $T^m = 0$, then $\sigma(T^m) = \{0\}$ and so $\sigma(T) = \{0\}$.

5. Let $T \in B(X)$. Show that $\exp T$ is a well defined bounded linear operator on X with

$$\|\exp T\| \leq \exp \|T\|$$

Solution: Note that

$$\exp T = \sum_{j=0}^{\infty} \frac{1}{j!} T^j.$$

Taking norms gives

$$\|\exp T\| = \left\| \sum_{j=0}^{\infty} \frac{1}{j!} T^j \right\| \leq \sum_{j=0}^{\infty} \frac{1}{j!} \|T\|^j = \exp \|T\|.$$

6. Show that the spectral radius depends upper semicontinuously in the norm topology. Namely if $T_n \rightarrow T$ then

$$\overline{\lim} r_{\sigma}(T_n) \leq r_{\sigma}(T).$$

Solution: Suppose that $\lambda \in \rho(T)$. Then $T_{\lambda} = T - \lambda I$ is invertible, has a bounded inverse and is densely defined. Since the set of invertible operators is open, and since $T_n \rightarrow T$ in norm we have that $\lambda \in \rho(T_n)$. This implies that $\rho(T) \subset \rho(T_n)$, or equivalently $\sigma(T_n) \subset \sigma(T)$. Which gives

$$r_{\sigma}(T_n) \leq r_{\sigma}(T)$$

and

$$\overline{\lim} r_{\sigma}(T_n) \leq r_{\sigma}(T).$$

7. If X is a complex Banach space, $S, T \in B(X)$ and $ST = TS$, show that

$$r_{\sigma}(ST) \leq r_{\sigma}(S)r_{\sigma}(T)$$

Solution: We have that

$$(ST)^2 = (ST)(ST) = (TS)(ST) = (TS)(TS) = T(ST)S = T(TS)S = T^2S^2.$$

Induction then proves

$$(ST)^k = T^k S^k$$

and so

$$\|(ST)^k\| \leq \|T^k\| \|S^k\|.$$

This inequality then clearly implies the result.

8. Show the commutativity can not be dropped from the problem above.

Solution: One can construct lots of different examples. Using 2x2 matrices is good place to start looking for an example. Let

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $r_\sigma(S) = r_\sigma(T) = 1$. Now compute that

$$ST = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad TS = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

So $ST \neq TS$. A simple computation then gives that the eigenvalues of ST are given by $2 \pm \sqrt{3}$. So we have that $r_\sigma(ST) = 2 + \sqrt{3} > r_\sigma(S)r_\sigma(T) = 1$.

9. Let H be a Hilbert space over $\mathbb{C}$. Show

- (a) If U is a bounded unitary operator and $\lambda \in \sigma(U)$, then $|\lambda| = 1$;
- (b) If S is a bounded self-adjoint and $\lambda \in \sigma(S)$, then $\lambda \in \mathbb{R}$.

Solution: For part (a) first note that $U^*U = UU^* = I$ and so $\|U\| = 1$. If $|\lambda| > 1$ then we have $\|\frac{U}{\lambda}\| < 1$ and thus $\lambda I - U = \lambda(I - \lambda^{-1}U)$ is invertible. So we have that $\lambda \notin \sigma(U)$. If $|\lambda| < 1$, then we have $\|\lambda U^*\| < 1$ and then $\lambda I - U = U(\lambda U^* - I)$ is invertible, and so $\lambda \notin \sigma(U)$. Thus we have that $\sigma(U) \subset \{\lambda : |\lambda| = 1\}$.

For part (b) note that if λ is an eigenvalue with eigenvector x , then we have

$$\lambda \langle x, x \rangle = \langle Ax, x \rangle = \langle x, A^*x \rangle = \langle x, Ax \rangle = \bar{\lambda} \langle x, x \rangle.$$

This gives that $\lambda = \bar{\lambda}$ for eigenvalues, and so must be real. Now observe that $\lambda \in \sigma(T)$ if and only if $\bar{\lambda} \in \sigma(T^*)$. But since $T = T^*$, then we have that $\lambda \in \sigma(T)$ if and only if

$\bar{\lambda} \in \sigma(T)$. We just need to rule out the case of the residual spectrum. So if $\lambda = a + ib$ then we have

$$\|(T - \lambda I)x\|^2 = \|(T - aI)x\|^2 + b^2 \|x\|^2 \geq b^2 \|x\|^2.$$

This estimate implies that $T - \lambda I$ is one-to-one and has closed range. If $\text{Ran}(T - \lambda I) \neq H$, then λ belongs to the residual spectrum. But, then we must have that $\bar{\lambda} = a - bi$ must be an eigenvalue of T , which is not real. This contradicts what we had from before.

10. Let $X = C[0, 1]$ with the standard norm. Define the operator

$$V(f)(t) = \int_0^t f(x) dx$$

Show that $\sigma(V) = \{0\}$ by two different methods:

- (a) Compute the spectral radius of V ;
- (b) Explicitly compute the inverse of the operator $V - \lambda I$ when $\lambda \neq 0$.

Solution: For part (a), note that

$$V^2(f)(t) = \int_0^t \left(\int_0^s f(u) du \right) ds$$

and so

$$|V^2(f)(t)| \leq \|f\|_X \int_0^t \left(\int_0^s du \right) ds = \frac{t^2}{2} \|f\|_X$$

So we have that $\|V^2\| \leq \frac{1}{2}$. A similar computation shows that

$$\|V^k\| \leq \frac{1}{k!}.$$

In fact one can easily show that

$$V^k f(s) = \frac{1}{(k-1)!} \int_0^s (s-t)^{k-1} f(t) dt.$$

These computations easily imply that

$$r_\sigma(V) = \lim_{k \rightarrow \infty} \|V^k\|^{\frac{1}{k}} = 0.$$

And so we have that $\sigma(T) = \{0\}$.

For part (b), we set up an ODE that we can solve. For the moment, assume that $y \in C[0, 1]$ is actually differentiable. We want to solve the equation

$$Vx(t) - \lambda x(t) = y(t) \quad t \in [0, 1].$$

This is the same as the ODE:

$$x'(t) - \frac{1}{\lambda}x(t) = -\frac{1}{\lambda}y'(t).$$

Set $\mu(t) = e^{-\frac{1}{\lambda}t}$, and then note that we have

$$\begin{aligned} x(t) = -\frac{1}{\lambda\mu(t)} \int_0^t y'(s)\mu(s) ds &= -\frac{1}{\lambda\mu(t)} \left(\int_0^t (y(s)\mu(s))' ds - \int_0^t y(s)\mu'(s) ds \right) \\ &= -\frac{1}{\lambda\mu(t)} \left(y(t)\mu(t) - y(0)\mu(0) + \frac{1}{\lambda} \int_0^t e^{-\frac{1}{\lambda}s} y(s) ds \right) \\ &= -\frac{1}{\lambda}y(t) + \frac{1}{\lambda}e^{\frac{1}{\lambda}t}y(0) - \frac{1}{\lambda^2} \int_0^t y(s)e^{\frac{1}{\lambda}s} ds. \end{aligned}$$

So we have that given $y \in C[0, 1]$ that its solution is given by

$$x(t) = -\frac{1}{\lambda}y(t) + \frac{1}{\lambda}e^{\frac{1}{\lambda}t}y(0) - \frac{1}{\lambda^2} \int_0^t y(s)e^{\frac{1}{\lambda}s} ds.$$

It is clear from this representation that we didn't need to apriori assume that y was differentiable.