Homework 4 Solutions

1. (a) The interval of "up 67 cents" corresponds to the ratio

$$2^{67/100}$$

which given

as a radical is	$\sqrt[1200]{2^{67}}$
as a power of 2 is	$2^{\frac{67}{100}}$
as a decimal is	≈ 1.039

(b) "Down 1050 cents" is given by

$$\frac{1}{\sqrt[1200]{2^{1050}}} = \frac{1}{\sqrt[8]{2^7}} \text{ or } 2^{\frac{-1050}{1200}} = 2^{-7/8} \text{ or } \approx 0.545$$

(c) "Up a major sixth" means going up nine semitones and this has ratio

$$\sqrt[12]{2^9} = \sqrt[4]{2^3} \text{ or } 2^{3/4} \text{ or } \approx 1.682$$

(d) The interval from B_3 to $G_1^\sharp$ is down two octaves plus a minor third (or 27 semitones) and in ratio form this is

$$\frac{1}{\sqrt[12]{2^{27}}} = \frac{1}{\sqrt[4]{2^9}} \text{ or } 2^{-9/4} \text{ or } \approx 0.210$$

2. Assume $A_4 = 440$Hz.

(a) Since C_4 is down a major sixth (or down 9 semitones) from A_4 the frequency is

$$C_4 = 440 \cdot 2^{-9/12} = 440 \cdot 2^{-3/4} \approx 261.6\text{Hz}$$

Applying similar reasoning for the rest of the problems, I get the following frequencies

(b) $D_2^\sharp = 440 \cdot 2^{-30/12} \approx 77.8$Hz

(c) $F_3 = 440 \cdot 2^{-4/3} \approx 174.6$Hz

(d) $E_1^\flat 440 \cdot 2^{-7/2} \approx 38.9$Hz

Now, assuming C_4 is tuned to 256Hz I get the following frequencies

(e) $\mathrm{A}_4 = 256 \cdot 2^{3/4} \approx 430.5$Hz

(f) $\mathrm{G}^\flat_6 = 256 \cdot 2^{30/12} \approx 1448.2$Hz

(g) $\mathrm{C}_1 = 256 \cdot 2^{-3} = 32$Hz

(h) $\mathrm{F}^\sharp_2 = 256 \cdot 2^{-3/2} \approx 90.5$Hz

3. Chord note frequencies (assuming $\mathrm{A}_4 = 440$Hz):

   (a) The notes of the major triad with root E_3 are $\mathrm{E}_3, \mathrm{G}^\sharp_3, \mathrm{B}_3$ and so the frequencies in Hertz are

   $$440 \cdot 2^{-17/12}, \qquad 440 \cdot 2^{-13/12}, \qquad 440 \cdot 2^{-10/12}$$

   or approximately $164.8, 207.7, 246.9$

   (b) Here we have $\mathrm{F}^\sharp_4, \mathrm{A}_4, \mathrm{C}^\sharp_5$ and the pitches are

   $$440 \cdot 2^{-3/12}, \qquad 440, \qquad 440 \cdot 2^{4/12}$$

   or approximately $370.0, 440, 554.4$ in Hertz.

   (c) The minor seventh with root A_5 is $\mathrm{A}_5, \mathrm{C}_6, \mathrm{E}_6, \mathrm{G}_6$ and the corresponding pitches are

   $$880, \quad 880 \cdot 2^{3/12}, \quad 880 \cdot 2^{7/12}, \quad 880 \cdot 2^{10/12}$$

   or approximately $880, 1046.5, 1318.5, 1568.0$ in Hertz.

   (d) The diminished seventh with root $\mathrm{A}^\flat_3$ is $\mathrm{A}^\flat_3, \mathrm{C}^\flat_3, \mathrm{D}_4, \mathrm{F}_4$ and the corresponding pitches are

   $$220 \cdot 2^{-1/12}, \quad 220 \cdot 2^{2/12}, \quad 220 \cdot 2^{5/12}, \quad 220 \cdot 2^{8/12}$$

   or approximately $207.7, 246.9, 293.7, 349.2$ in Hertz.

4. Recall that if you have a string of length L and frequency f with a fixed "tautness", then to make a frequency with ratio r to f the length of the string should be changed from L to L/r. Therefore, the ratio corresponding to x semitones (namely $2^{x/12}$) would require a length of string equal to $L \cdot 2^{-x/12}$. So, the first twelve frets should be placed at the following locations from the "bridge" of the banjo (i.e. the bottom of the string).

semitones	1	2	3	4
distance (cm)	$40 \cdot 2^{-1/12}$	$40 \cdot 2^{-2/12}$	$40 \cdot 2^{-3/12}$	$40 \cdot 2^{-4/12}$
approx ≈	37.8	35.6	33.6	31.7

semitones	5	6	7	8
distance (cm)	$40 \cdot 2^{-5/12}$	$40 \cdot 2^{-6/12}$	$40 \cdot 2^{-7/12}$	$40 \cdot 2^{-8/12}$
approx ≈	30.0	28.3	26.7	25.2

semitones	9	10	11	12
distance (cm)	$40 \cdot 2^{-9/12}$	$40 \cdot 2^{-10/12}$	$40 \cdot 2^{-11/12}$	$40 \cdot 2^{-1}$
approx ≈	23.8	22.4	21.2	20

5. Fun with logarithms:

   (a)
   $$\log_{10}(0.01) = \log_{10}(10^{-2}) = -2$$

   (b)
   $$\log_2 16 = \log_2 2^4 = 4$$

   (c)
   $$\log_5 \sqrt[3]{25} = \log_5 5^{2/3} = \frac{2}{3}$$

   (d)
   $$\log_c \sqrt[n]{c^l} = \log_c c^{l/n} = \frac{l}{n}$$

   (e)
   $$\log_2 5 + \log_2 3 = \log_2(5 \cdot 3) = \log_2(15)$$

   (f)
   $$\log_4 7 - 2\log_4 11 = \log_4 7 - \log_4 11^2 = \log_4(7/121)$$

   (g) Observe that
   $$\begin{aligned}\log_9 16 &= \frac{\ln 16}{\ln 9} = \frac{\ln 16}{\ln 3^2} = \frac{\ln 16}{2\ln 3} \\ &= \frac{1}{2}\log_3 16 = \log_3 \sqrt{16} = \log_3 4\end{aligned}$$

   and therefore
   $$\log_3 10 + \log_9 16 = \log_3 10 + \log_3 4 = \log_3(40)$$

(h) Notice that

$$2\log_a x^2 = \log_a x^4$$

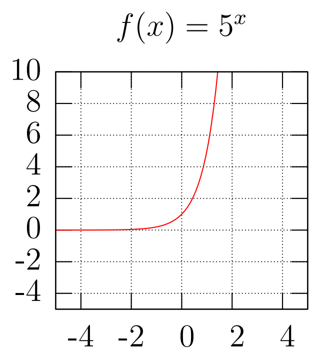
and

$$\frac{1}{2}\log_{\sqrt{a}} x = \frac{\ln x}{2\ln\sqrt{a}} = \frac{\ln x}{\ln a} = \log_a x$$

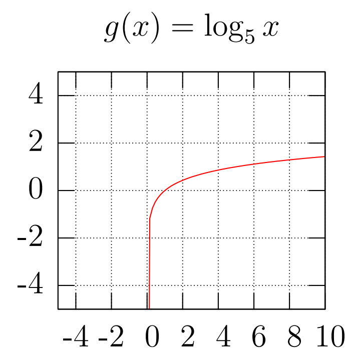
which yields

$$2\log_a x^2 - \frac{1}{2}\log_{\sqrt{a}} x = \log_a x^4 - \log_a x = \log_a(x^4/x) = \log_a x^3$$

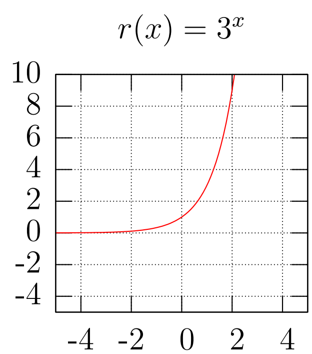
6. Graphs:



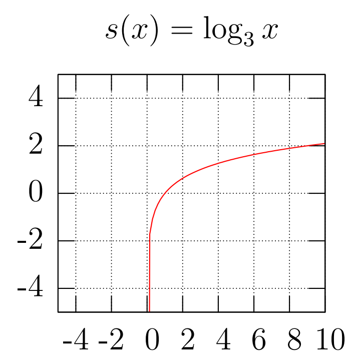
(a)



(b)



(c)



(d)

The functions f and g are inverse to each other; as are r and s. The functions f and r are horizontal “dilates” (or stretches) of one another since

$$f(x) = 5^x = (3^{\log_3 5})^x = 3^{(\log_3 5)x} = r(x\log_3 5)$$

i.e. if you stretch r horizontally by a factor of $(\log_3 5)^{-1}$ you get f. The functions g and s are vertical stretches of one another since

$$g(x) = \log_5 x = \log_5 3^{\log_3 x} = (\log_5 3) \log_3 x = (\log_5 3)s(x)$$

i.e. if you stretch s by a factor of $\log_5 3$ (vertically) you get g. Notice (uncoincidentally) that $(\log_3 5)^{-1} = \log_5 3$.

7. **Claim:** $\log_b(x/y) = \log_b x - \log_b y$

   Proof. By definition of logarithms, $\log_b(x/y)$ is the (unique) power to which we must raise b in order to obtain x/y. So, if we can show

   $$b^{\log_b x - \log_b y} = \frac{x}{y}$$

   then the claim will be established. This is not too hard though:

   $$b^{\log_b x - \log_b y} = \frac{b^{\log_b x}}{b^{\log_b y}} = \frac{x}{y}$$

   by properties of exponents and the fact that logarithms and exponentials are inverses of each other (when they have the same base). □

8. If we have a logarithmic scale for which a distance of n corresponds to an octave, then a distance of 1 corresponds to a ratio r with the property that $r^n = 2 =$ octave ratio. If we take n-th roots, we see that $r = \sqrt[n]{2}$. The base we should choose is the same as $\sqrt[n]{2}$ because if $\log_b \sqrt[n]{2} = 1$ (this is just saying our earlier requirement that $\sqrt[n]{2}$ corresponds to a distance of 1) then it is automatic that $b = \sqrt[n]{2}$, because if we raise everything to the b power we get

   $$b = b^{\log_b \sqrt[n]{2}} = \sqrt[n]{2}$$

9. Recall that a ratio r correponds to $12 \log_2 r$ semitones.

   (a) So, a ratio of 3 corresponds to $12 \log_2 3 \approx 19.02$ semitones. We apply the same reasoning for the rest of the answers.

   (b) $12 \log_2 0.8 \approx -3.86$ semitones

   (c) $12 \log_2(4/3) \approx 4.98$ semitones

(d) $12 \log_2 \sqrt[3]{2} \approx 4$ semitones

(e) $12 \log_2 e \approx 17.31$ semitones

Next, we convert ratios into cents via the similar formula cents $= 1200 \log_2 r$.

(f) For example, a ratio of 1.25 corresponds to $1200 \log_2 1.25 \approx 386$ cents. We apply the same formula for the rest of the answers.

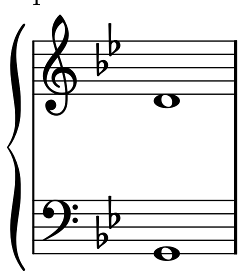
(g) $1200 \log_2 1.1 \approx 165$ cents

(h) $1200 \log_2(7/4) \approx 969$ cents

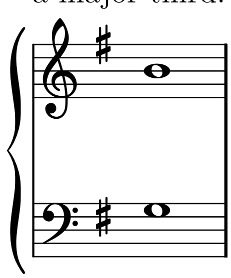
(i) $1200 \log_2(2/3) \approx -702$ cents

(j) $1200 \log_2 \pi \approx 1982$ cents

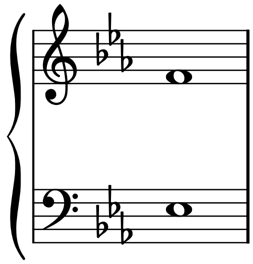
10. (a) A ratio of 3 is closest to 19 semitones ($\approx 12 \log_2 3$) or an octave plus a fifth:



(b) A ratio of $2/5$ is closest to -16 semitones or down an octave plus a major third:



(c) A ratio of 2.3 corresponds to roughly 14 semitones or an octave plus a step:



(d) A ratio of π^{-1} corresponds to roughly -20 semitones or down an octave plus a minor sixth: