EXAM I
Math 109 / Music 109A, Spring 2018

Name ________________________________ Id ____________________

Each problem is worth 10 points.

1. Sketch the graphs of these functions by starting with a more basic function and applying one or more geometric transformations (shifts or stretches). Use the space on page 4 if you need it.
   
   (a) \( f(x) = -x^2 + 2 \)
   
   (b) \( g(x) = -1 + \cos \frac{x}{4} \)

2. For the following pairs of integers \( m, n \), find the numbers \( q \) and \( r \) whose existence is asserted in the division algorithm \( n = qm + r \):

   (a) \( 7, -22 \);
   
   \[ -22 = -4 \cdot 7 + 6 \]
   
   \[ q = -4, \quad r = 6 \]

   (b) \( 3, 102l + 4 \), where \( l \) some integer.
   
   \[ 102l + 4 = (34l + 1) \cdot 3 + 1 \]
   
   \[ q = 34l + 1, \quad r = 1 \]

3. Write the indicated note as a whole note, choosing and notating an appropriate clef.

   (a) \( \text{\begin{tikzpicture}
       \draw[thick] (0,0) -- (1,0);
       \draw[thick] (0.25,-0.25) -- (0.25,0.25);
       \draw[thick] (0.5,-0.5) -- (0.5,0.5);
       \draw[thick] (0.75,-0.75) -- (0.75,0.75);
       \draw[thick] (1,1) -- (1,-1);
     \end{tikzpicture}} \)

   \( F_4 \)

   (b) \( \text{\begin{tikzpicture}
       \draw[thick] (0,0) -- (1,0);
       \draw[thick] (0.25,0) -- (0.25,-0.25);
       \draw[thick] (0.5,0.5) -- (0.5,-0.5);
       \draw[thick] (0.75,0.75) -- (0.75,-0.75);
       \draw[thick] (1,1) -- (1,-1);
     \end{tikzpicture}} \)

   \( A_5^\sharp \)

   (c) \( \text{\begin{tikzpicture}
       \draw[thick] (0,0) -- (1,0);
       \draw[thick] (0.25,0) -- (0.25,0.25);
       \draw[thick] (0.5,0.5) -- (0.5,0.75);
       \draw[thick] (0.75,0.75) -- (0.75,0.5);
       \draw[thick] (1,0) -- (1,0.75);
     \end{tikzpicture}} \)

   \( B_6^\flat \)
4. For the set \( \{(a, b) \in \mathbb{Z}^2 \mid b \neq 0\} \) show that the relation \( \sim \) defined by \( (a, b) \sim (a', b') \) iff \( ab' - a'b = 0 \) is an equivalence relation. Explain how the set of equivalence classes are in one-to-one correspondence with the set of rational numbers \( \mathbb{Q} \).

OR

For the set \( \mathbb{Z} \) and a fixed positive integer \( m \), show that the relation \( \equiv \) defined by \( k \equiv \ell \) iff \( m \mid (k - \ell) \) is an equivalence relation. Explain why there are exactly \( m \) equivalence classes.

\[(a, b) \sim (a', b') \text{ means } a'b' - a'b = 0 \text{, or equivalently } a b' = a' b \]

reflexive: \( \frac{a}{b} = \frac{a}{b} \) so \( (a, b) \sim (a, b) \)

symmetric: Assume \( (a, b) \sim (a', b') \). Then \( \frac{a}{b} = \frac{a'}{b'} \) so \( a' = \frac{a}{b} b' \)

hence \( (a, b') \sim (a, b) \)

transitive: Assume \( (a, b) \sim (a', b') \) and \( (a', b') \sim (a'', b'') \). Then \( \frac{a'}{b'} = \frac{a''}{b''} \) and \( \frac{a}{b} = \frac{a'}{b'} \). So \( \frac{a}{b} = \frac{a''}{b''} \) hence \( (a, b) \sim (a'', b'') \).

The function that sends the class \( [(a, b)] \) to \( \frac{a}{b} \) gives a 1-1 correspondence between classes and elements of \( \mathbb{Q} \).

reflexive: \( k \equiv k \) so \( k \equiv k \).

symmetric: If \( k \equiv \ell \), then \( k - \ell = am \). Then \( k - \ell = (a \cdot m) \).

so \( k \equiv \ell \).

transitive: Assume \( k \equiv \ell \) and \( \ell \equiv t \). Then \( k - \ell = am \), \( \ell - t = bm \). Adding these equalities gives \( k - t = (a + b)m \).

so \( k \equiv t \).

Claim: \( \mathbb{Z}/m\mathbb{Z} \) is the set of \( 0 \) through \( m-1 \) congruent classes.

5. Add the needed sharps or flats to notes so that the following gives the Lydian scale tones \( 1 \) to \( 8 \), from \( D \) to \( D \). (Do not alter \( D \); do not write in a key signature.)

![Lydian scale](image-url)

\( \text{1 2 3 4 5 6 7 8} \)

2
6. For the following modes and tonic notes, indicate the appropriate key signature on the given staff, taking note of the clef:

(a) Phrygian with tonic D

(c) Aeolian with tonic G♯

7. Identify each chord in this major mode (Ionian) passage. Above the staff label each chord by root note class with suffix (e.g., B⁷). Below the staff, label each chord by root scale tone (e.g. bIII⁷).

8. Extend the following melody with two measures having the same rhythm, employing the following transformations. Do not write in a key change.

(a) diatonic up two scale tones in the second measure

(b) chromatic up a major third (from the original) in the third measure
9. Give the (total) duration in beats of:

(a) a doubly-dotted quarter note in $\frac{3}{8}$ time.

\[
\text{quarter note has duration } \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16} \text{ of a whole note. So } 2 \times \frac{3}{16} = \frac{3}{8} \text{ beats.}
\]

(b) a sixteenth note in $\frac{5}{8}$ time (compound time signature).

\[
\text{So } \frac{1}{6} \text{ of a whole note. So } 2 \times \frac{1}{6} = \frac{2}{3} \text{ beats.}
\]

(c) a quarter note quintuplet in $\frac{3}{4}$ time.

\[
\frac{1}{4} \times \frac{3}{5} = \frac{3}{20} \text{ of a whole note. So } 4 \times \frac{3}{20} = \frac{3}{5} \text{ beats.}
\]

10. For the song *Mary Had A Little Lamb*, give the form (e.g., AABC) by dividing it into segments consisting of two bars. Locate and identify a translation other than that which comes from the overall form.

![Musical notation of Mary Had A Little Lamb](image)

**Translation:**

- *A* to *A*
- *B* to *A* (harmonic translation)
- *C* to *B* (rhythmic translation)