EXAM III

Math 109 / Music 109A, Spring 2018

Name Solutions Id

Each problem is worth 10 points.

1. Define a (commutative) ring $R$ and show that $0 \cdot x = 0$, for any $x \in R$. Use this to show that $(-1) \cdot x = -x$, for any $x \in R$.

A ring $R$ is a set with two laws of composition $+$ and $\cdot$, such that $(R, +)$ is a commutative group and $(R, \cdot)$ is a commutative monoid. In addition, for all $a, b, c \in R$ we have $a \cdot (b + c) = a \cdot b + a \cdot c$.

$$0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x.$$ Subtracting $0 \cdot x$, we get $0 = 0 \cdot x$.

To see that $(-1) \cdot x$ is the additive inverse of $x$, we write $(-1) \cdot x + x = (-1) \cdot x + 1 \cdot x = (-1 + 1) \cdot x = 0 \cdot x = 0$. So $(-1) \cdot x = -x$.

2. Determine whether these subsets of $\mathbb{Z}$ are ideals. If so, express them in the form $n\mathbb{Z}$, where $n$ is a positive integer. Justify your answers.

(a) the positive integers

No. Does not contain $0$, nor negatives.

(b) $100\mathbb{Z} + 30\mathbb{Z}$

gcd $(100, 30) = 10$

so $10\mathbb{Z}$
3. Prove that there are infinitely many prime numbers.

Assume not. Then there are only finitely many primes \( p_1, \ldots, p_r \). Let \( N = p_1 \cdots p_r + 1 \). Then \( N \) has a prime factor \( p_i \), so \( p_i \mid N \), i.e., \( p_i \mid (p_1 \cdots p_r + 1) \). But \( p_i \mid p_1 \cdot p_r \), so we must have \( p_i \mid 1 \), a contradiction.

4. List the units of the ring \( \mathbb{Z}_{12} \), i.e., the elements of \( \mathbb{Z}_{12}^* \), and explain the implication this has for keyboard intervals.

\[ \mathbb{Z}_{12}^* = \{ 1, 5, 7, 11 \} \] The corresponding keyboard intervals are the semitones, fourths, fifths, and major sevenths. These are the generating intervals, i.e., those which, when iterated, give all note classes.

5. Give the prime factorizations of these integers, writing the primes in ascending order, as in \( 2^3 \cdot 3^1 \cdot 7^2 \).

(a) 75 \hspace{1cm} (b) 64 \hspace{1cm} (c) 242 \hspace{1cm} (d) 52 \hspace{1cm} (e) 14 \times 10^{23}

\begin{align*}
75 &= 3^1 \cdot 5^2 \hspace{1cm} 64 &= 2^6 \hspace{1cm} 242 &= 2 \cdot 11^2 \hspace{1cm} 52 &= 2^2 \cdot 13 \hspace{1cm} 14 \times 10^{23} &= 2^4 \cdot 5^{23} \cdot 7^1
\end{align*}
6. Prove that if \( y = f(t) \) has period \( P \), then \( y = cf(t) \) has period \( P \), but \( y = f(ct) \) \((c \neq 0)\) has period \( P/c \).

We have \( f(t+P) = f(t) \)

Then \( cf(t+P) = cf(t) \) so \( y = cf(t) \) has period \( P \).

Also \( f(c(t+P)) = f(ct+P) = f(ct) \), so \( y = f(ct) \) has period \( P/c \).

7. On the staff system below, write the keyboard's best approximation for harmonics 1 through 13 for the indicated note. For the 7th, 11th, and 13th harmonics, indicate how sharp or flat (to the nearest cent) the keyboard's approximation is.

\[
\begin{align*}
12\log_{2}7 & \approx 33.69 \\
& \text{so keyboard is } \approx 31 \text{ cents sharp}
\end{align*}
\]

\[
\begin{align*}
12\log_{2}11 & \approx 41.51 \\
& \text{so keyboard is } \approx 49 \text{ cents sharp}
\end{align*}
\]

\[
\begin{align*}
12\log_{2}13 & \approx 44.41 \\
& \text{so keyboard is } \approx 44 \text{ cents flat}
\end{align*}
\]

8. Find the value \( \gamma \) for which the pitch associated to the periodic function
\( h(t) = d \sin(\gamma t + \beta) \), where \( t \) is time in seconds, is \( E_4^\# \).

\[
\begin{align*}
\gamma &= \frac{2\pi F}{2} \\
&= \frac{2\pi \cdot 440 \cdot 2^{-6/12}}{} \\
&\approx 19.54, 87
\end{align*}
\]
9. Find the period, frequency, amplitude, and phase shift for the function

\[ g(t) = 3 \sin(880\pi t) + 4 \cos(880\pi t) \]

and express it in the form \( d \sin(\alpha t + \beta) \), giving a decimal approximation for \( \beta \).

\[
\begin{align*}
\text{amplitude} & = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \\
\text{frequency} & = \frac{880\pi}{2\pi} = 440 \text{ Hz}
\end{align*}
\]

\[ g(t) = 5 \left[ \frac{3}{5} \sin(880\pi t) + \frac{4}{5} \cos(880\pi t) \right] \]

\[ \text{\( \cos \beta = \frac{3}{5} \) \quad \sin \beta = \frac{4}{5} \]

\[ g(t) = 5 \sin(880\pi t + \beta) \]

\[ \beta = \arcsin \frac{4}{5} \approx 0.93 \]

10. A certain vowel sound has a formant which amplifies frequencies within 350 Hz of 2900 Hz. An alto singer sings the vowel at \( A_3 \). Which harmonics are amplified?

\[ A_3 = 220 \text{ Hz} \]

\[ 2550 \leq 220k \leq 3250 \]

\[ \frac{2550}{220} \leq k \leq \frac{3250}{220} \]

\[ 11.59 \leq k \leq 14.77 \]

\( k = 12, 13, 14 \)