FINAL EXAM

Math 109 / Music 109A, Spring 2018

Name Solutions Id ____________

Each problem is worth 10 points. Round off each decimal approximation to two digits to the right of the decimal. Your exam should be returned via Crowdmark by Wednesday, May 2. If this timeline is problematic please contact the professor.

1. Give the (total) duration in beats of:

   (a) a quarter note in $\frac{12}{8}$ time (compound time signature). $\boxed{\frac{2}{3} \text{ beat}}$

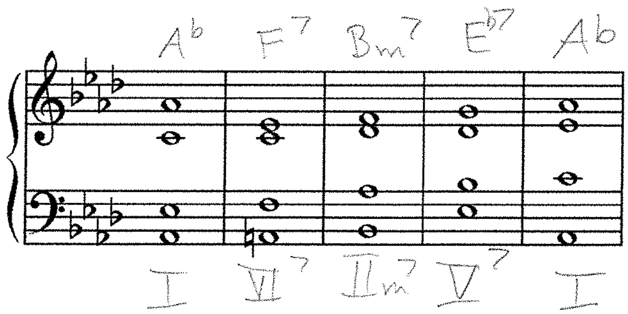
   (b) a dotted eighth note in $\frac{3}{2}$ time. 𝅗𝅥 $= 1$, ♩ $= \frac{1}{2}$, ♪ $= \frac{1}{4}$, ♪. $= \frac{1}{4}(1+\frac{1}{2}) = \frac{1}{4}\cdot\frac{3}{2} = \boxed{\frac{3}{8}}$

   (c) a quarter note 7-tuplet in $\frac{4}{4}$ time. $2^2 < 7 < 2^3$ so $r = 2$

   $n + 2 = 2$ so $n = 0$.

   $\frac{1}{2^0}$-note = whole note. $\boxed{4 \text{ beats}}$

2. Identify each chord in this major mode (Ionian) passage. Above the staff label each chord by root note class with suffix (e.g., $E^{\flat 7}$). Below the staff, label each chord by root scale tone (e.g. $\flat III^7$).



3. Convert to semitones the musical intervals given by the following ratios, indicating whether the interval is upward or downward.

(a) 0.9 $\quad 12\log_2(0.9) \approx -1.92 \quad$ $\boxed{\text{1.82 semitones downward}}$

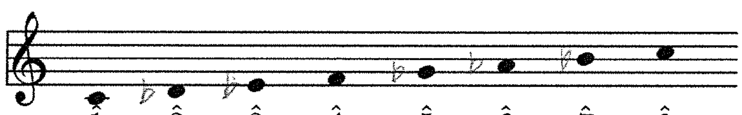
(b) π $\quad 12\log_2(\pi) \approx 19.82 \quad$ $\boxed{\text{19.82 semitones upward}}$

Express as a ratio the following musical intervals.

(c) 137 cents $\quad 2^{137/100} \approx \boxed{1.08}$

(d) the just minor third $\quad \boxed{\frac{6}{5} = 1.2}$

4. Add the needed sharps or flats <u>to notes</u> so that the following gives the <u>Locrian</u> scale tones $\hat{1}$ to $\hat{8}$, from C to C. Do not alter C, do not write in a key signature.



$\hat{1}$ $\hat{2}$ $\hat{3}$ $\hat{4}$ $\hat{5}$ $\hat{6}$ $\hat{7}$ $\hat{8}$

5. Suppose a 12-tone row chart begins: B, D^b, A^b, E, G, ... Write the upper left 5×5 matrix of the resulting row chart. Then rewrite it replacing each note class with the element of $\mathbb{Z}_{12}$ which measures its modular interval from B.

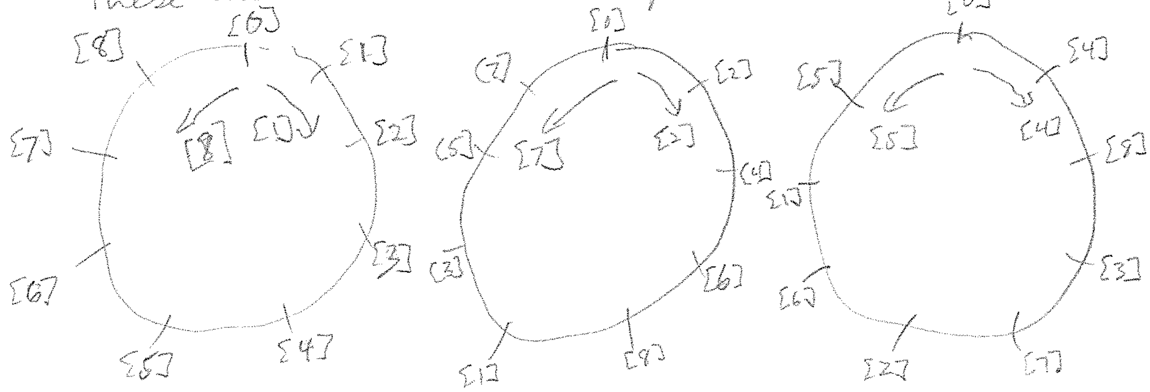
B	D^b	A^b	E	G
A	B	G^b	D	F
D	E	B	G	B^b
G^b	A^b	E^b	B	D
E^b	F	C	A^b	B

[0]	[2]	[9]	[5]	[8]
[10]	[0]	[7]	[3]	[6]
[3]	[5]	[0]	[8]	[11]
[7]	[9]	[4]	[0]	[3]
[4]	[6]	[1]	[9]	[0]

[1], [8] inverses
[2], [7] inverses
[4], [5] inverses

6. If the keyboard divided the octave into 9 rather than 12 equal intervals, what would be the generating intervals? Quote the relevant facts from modular arithmetic that justify your answer. Draw the circles that exhibit inverse pairs of these generating intervals.

The generators of $\mathbb{Z}_9$ are [1], [2], [4], [5], [7], [8]
These are the classes [m] for which gcd(m, 9) = 1.



7. (a) Find the period, frequency, amplitude, and phase shift for the function

$$g(t) = \sin(660\pi t) + \sqrt{3}\cos(660\pi t)$$

and express it in the form $d\sin(\alpha t+\beta)$, giving a decimal approximation for β. Identify the closest keyboard note to the pitch represented by this function, and the error (if any) in cents.

$d = \sqrt{1^2+\sqrt{3}^2} = \sqrt{1+3} = \sqrt{4} = \boxed{2 = \text{amplitude}}$

$g(t) = 2\left[\frac{1}{2}\sin(660\pi t) + \frac{\sqrt{3}}{2}\cos(660\pi t)\right]$

$= \boxed{2\sin(660\pi t + \beta)}$ where $\beta = \boxed{\arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3} \approx 1.05 \text{ phase shift}}$

$660\pi = 2\pi F$ so $\boxed{F = 330 \text{ hz frequency}}$ $\quad P = \boxed{\frac{1}{330} \approx 0.0030 \text{ period}}$

Measuring from $A_3 = 220$,

$\frac{330}{220} = \frac{3}{2} \qquad 12\log_2\left(\frac{3}{2}\right) \approx 7.02$ semitones

$\boxed{E_4 \text{ is the closest keyboard note, 2 cents under } F = 330.}$

(b) Find the period, frequency, amplitude, and phase shift for the function

$$h(t) = 3\sqrt{2}\sin\left(1500t + \frac{\pi}{4}\right)$$

and express it in the form $A\sin\alpha t + B\cos\alpha t$. Identify the closest keyboard note to the pitch represented by this function, and the error (if any) in cents. Caution: It says $1500t$, not $1500\pi t$.

$d = \boxed{3\sqrt{2} \text{ amplitude}}$ $\quad \beta = \boxed{\frac{\pi}{4} \text{ phase shift}}$

$2\pi F = 1500 \quad$ So $F = \frac{1500}{2\pi} \approx \boxed{238.73 \text{ hz frequency}}$

$P = \frac{1}{F} \approx \boxed{0.0042 \text{ period}}$

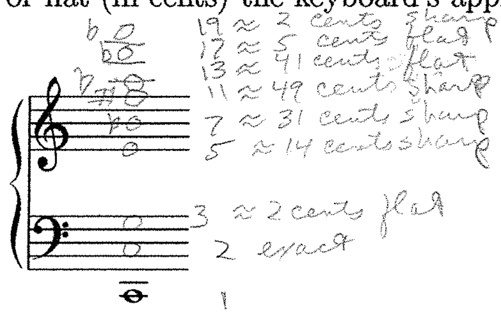
Measuring from $A_3 = 220$,

$12 \log_2\left(\frac{F}{220}\right) \approx 1.41$ $\quad$ $\boxed{\text{So } B^b_3 \text{ is closest keyboard note, } 41 \text{ cents below } F}$

$h(t) = 3\sqrt{2}\left[\overbrace{\cos\frac{\pi}{4}}^{1/\sqrt{2}}\sin(1500t) + \overbrace{\sin\frac{\pi}{4}}^{1/\sqrt{2}}\cos(1500t)\right]$

$= \boxed{3\sin(1500t) + 3\cos(1500t)}$

8. On the staff system below, write the keyboard's best approximation for each prime harmonic up through 19 for the indicated note, indicating how sharp or flat (in cents) the keyboard's approximation is.



$12 \log_2 3 \approx 19.02$

$12 \log_2 5 \approx 27.86$

$12 \log_2 7 \approx 33.69$

$12 \log_2 11 \approx 41.51$

$12 \log_2 13 \approx 44.41$

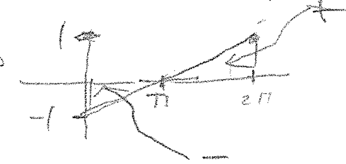
$12 \log_2 17 \approx 49.05$

$12 \log_2 19 \approx 50.98$

9. The sawtooth wave, defined on $[0, 2\pi)$ by $q(t) = \frac{1}{\pi}t - 1$, has Fourier series

$$q(t) = -\frac{2}{\pi}\left[\sin t + \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) + \frac{1}{4}\sin(4t) + \cdots\right]$$

Give the values of the Fourier coefficients C, A_k, B_k for $k \in \mathbb{Z}^+$, indicate the phase shift of each harmonic, and explain why the value of C is such, based on the graph of $q(t)$.



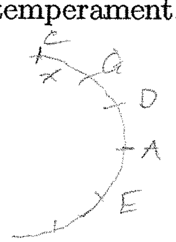
C = constant term = 0, which is reflected in cancelling areas between graph and x-axis.

$A_k = -\frac{2}{k\pi}$

Each phase shift $\beta_k = 0$.

$B_k = 0$ for all k.

10. (a) What is the mean-tone fifth? Give its value and explain how it arises. Compare it, in cents, to the just fifth and the fifth of equal temperament.



Mean tone 5^{th} x is chosen so that

$x^4 = 4 \cdot \frac{5}{4}$ (2 octaves + just 3rd)

$= 5$

so $x = 5^{1/4}$ $\quad 12\log_2(5^{1/4}) \approx 6.97$, so mean-tone fifth is ≈ 3 cents below tempered fifth

$12\log_2(5^{1/4}/(\frac{3}{2})) \approx -0.05$ so m.t. 5^{th} is ≈ 5 cents below just fifth $(\frac{3}{2})$.

(b) Compare three just thirds with an octave. Do the same for three Pythagorean thirds. In each case, is it greater than, less than, or equal to an octave, and by how much in cents?

Just 3rd is ratio $\frac{5}{4}$. 3 just 3rds is $\left(\frac{5}{4}\right)^3 = \frac{125}{64}$.

Since $\frac{125}{64} < \frac{128}{64} = 2$, it is less than an octave

In cents the difference is $1200\log_2(2/\frac{125}{64})$

$= 1200\log_2\left(\frac{128}{125}\right) \approx \boxed{41 \text{ cents}}$

Pythagorean 3rd is $(\frac{3}{2})^4/4 = \frac{3^4}{2^6} = \frac{81}{64}$

HAVE A GOOD SUMMER!

3 Pyth 3rds is $(\frac{81}{64})^3 \approx 2.03 > 2$ so greater than an octave.

In cents the difference is $1200\log_2[(\frac{81}{64})^3/2] \approx \boxed{23 \text{ cents}}$