FINAL EXAM
Math 109 / Music 109A, Spring 2018

Name Solutions Id

Each problem is worth 10 points. Round off each decimal approximation to two digits to the right of the decimal. Your exam should be returned via Crowdmark by Wednesday, May 2. If this timeline is problematic please contact the professor.

1. Give the (total) duration in beats of:

   (a) a quarter note in $\frac{12}{8}$ time (compound time signature).

   \[ \frac{2}{2} \text{ beat} \]

   (b) a dotted eighth note in $\frac{3}{2}$ time.

   \[ d = \frac{3}{4}, \quad d' = \frac{1}{2}, \quad d'' = \frac{1}{4} \quad \text{so} \quad \frac{3}{8} \text{ beat} \]

   (c) a quarter note 7-tuplet in $\frac{4}{4}$ time.

   \[ n + 2 = 2 \quad \text{so} \quad n = 0, \quad \frac{1}{2} \text{ notes} = \text{whole note} \quad \text{4 beats} \]

2. Identify each chord in this major mode (Ionian) passage. Above the staff label each chord by root note class with suffix (e.g., E$^\text{7}$). Below the staff, label each chord by root scale tone (e.g. bIII$^7$).

   \[ \text{Ab} \quad F^7 \quad Bm^7 \quad E^7 \quad Ab \]

   \[ I \quad \text{VII}^7 \quad \text{II}^7 \quad \text{IV}^7 \quad I \]
3. Convert to semitones the musical intervals given by the following ratios, indicating whether the interval is upward or downward.

(a) $0.9 \quad 12 \log_2 (0.9) \approx -1.92 \quad 1.82$ semitones downward

(b) $\pi \quad 12 \log_2 (\pi) \approx 19.82 \quad 19.82$ semitones upward

Express as a ratio the following musical intervals.

(c) 137 cents $\frac{137}{100} < \frac{1.08}{1}$

(d) the just minor third $\frac{6}{5} = 1.2$

4. Add the needed sharps or flats to notes so that the following gives the Locrian scale tones 1 to 8, from C to C. Do not alter C, do not write in a key signature.

![Musical notation image]

5. Suppose a 12-tone row chart begins: B, D♭, A♭, E, G, … Write the upper left 5 × 5 matrix of the resulting row chart. Then rewrite it replacing each note class with the element of $\mathbb{Z}_{12}$ which measures its modular interval from B.

<table>
<thead>
<tr>
<th>B</th>
<th>D♭</th>
<th>A♭</th>
<th>E</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>G♭</td>
<td>D</td>
<td>F</td>
</tr>
<tr>
<td>D</td>
<td>E</td>
<td>B</td>
<td>G♭</td>
<td>B♭</td>
</tr>
<tr>
<td>G♭</td>
<td>A♭</td>
<td>E♭</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>E♭</td>
<td>F</td>
<td>C</td>
<td>A♭</td>
<td>B</td>
</tr>
</tbody>
</table>

\[ \begin{array}{c|c|c|c|c|c} 2 & 12 & 9 & 6 & 3 & 0 \\ \hline 10 & 1 & 8 & 5 & 11 & 3 \end{array} \]
6. If the keyboard divided the octave into 9 rather than 12 equal intervals, what would be the generating intervals? Quote the relevant facts from modular arithmetic that justify your answer. Draw the circles that exhibit inverse pairs of these generating intervals.

The generators of \( \mathbb{Z}_9 \) are \([1], [2], [4], [5], [7], [8]\).
These are the classes \( \mathbb{Z}_9 \) for which \( \gcd(m, 9) = 1 \).

7. (a) Find the period, frequency, amplitude, and phase shift for the function

\[
g(t) = \sin(660\pi t) + \sqrt{3} \cos(660\pi t)
\]

and express it in the form \( d \sin(\alpha t + \beta) \), giving a decimal approximation for \( \beta \). Identify the closest keyboard note to the pitch represented by this function, and the error (if any) in cents.

\[
d = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2 = \text{amplitude}
\]

\[
g(t) = 2 \left( \frac{1}{2} \sin(660\pi t) + \frac{\sqrt{3}}{2} \cos(660\pi t) \right) = 2 \sin(660\pi t + \beta)
\]

where \( \beta = 60 \) degrees

660\pi = 20\pi \pm 20 \rightarrow F = 330 Hertz

Measuring from \( A_3 = 220 \),

\[
\frac{330}{220} = \frac{3}{2}
\]

12 \log_{2}\left(\frac{3}{2}\right) \approx 7.02 \text{ semitones}

\( E \) is the closest keyboard note, 2 cents
under \( F = 330 \).
(b) Find the period, frequency, amplitude, and phase shift for the function

\[ h(t) = 3\sqrt{2} \sin \left( 1500t + \frac{\pi}{4} \right) \]

and express it in the form \( A \sin \omega t + B \cos \omega t \). Identify the closest keyboard note to the pitch represented by this function, and the error (if any) in cents. Caution: It says 1500\(t\), not 1500\(\pi t\).

\[ 2\pi F = 1500 \quad \text{so} \quad F = \frac{1500}{2\pi} \approx 238.73 \text{ hz frequency} \]

\[ P = \frac{1}{F} \approx 0.0042 \text{ period} \]

Measuring from \( A_3 = 220, \)

\[ 12 \log_2 \left( \frac{F}{220} \right) \approx 1.41 \]

So B\# in closest keyboard note.

\[ h(t) = 3\sqrt{2} \left[ \cos \left( \frac{\omega}{2} \sin (1500t) \right) + \sin \left( \frac{\omega}{2} \cos (1500t) \right) \right] \]

\[ = 3 \cos (1500t) + 3 \cos (1500t) \]

8. On the staff system below, write the keyboard’s best approximation for each prime harmonic up through 19 for the indicated note, indicating how sharp or flat (in cents) the keyboard’s approximation is.

\[ \begin{align*}
1 & \approx 2 \text{ cents flat} \\
2 & \text{ exact} \\
3 & \approx 2 \text{ cents flat} \\
5 & \approx 14 \text{ cents sharp} \\
7 & \approx 31 \text{ cents sharp} \\
11 & \approx 47 \text{ cents sharp} \\
13 & \approx 59 \text{ cents flat} \\
17 & \approx 89 \text{ cents flat} \\
19 & \approx 119 \text{ cents flat}
\end{align*} \]
9. The sawtooth wave, defined on \([0, 2\pi]\) by \(q(t) = \frac{1}{\pi}t - 1\), has Fourier series
\[
q(t) = -\frac{2}{\pi} \left[ \sin t + \frac{1}{2} \sin(2t) + \frac{1}{3} \sin(3t) + \frac{1}{4} \sin(4t) + \cdots \right]
\]
Give the values of the Fourier coefficients \(C, A_k, B_k\) for \(k \in \mathbb{Z}^+\), indicate the phase shift of each harmonic, and explain why the value of \(C\) is such, based on the graph of \(q(t)\).

\[
C = \text{constant term} = 0, \quad \text{which is reflected in cancelling areas between graph and } x\text{-axis}.
\]
\[
A_k = -\frac{2}{k\pi}
\]
\[
B_k = 0 \quad \text{for all } k.
\]

10. (a) What is the mean-tone fifth? Give its value and explain how it arises. Compare it, in cents, to the just fifth and the fifth of equal temperament.

\[
\text{Mean tone } 5\text{th}\quad x \text{ is chosen so that}
\]
\[
x^4 = 4 \cdot \frac{5}{4} \quad (2\text{ octaves plus just 3rd})
\]
\[
x = 5^{1/4} \approx 1.0696
\]
\[
\text{mean tone fifth is } x^3 \text{ cents below tempered fifth}
\]
\[
12\log_x \left( \frac{5^{3/4}}{2} \right) \approx 0.05 \text{ so } 1.0696^3 \approx 5 \text{ cents}
\]

(b) Compare three just thirds with an octave. Do the same for three Pythagorean thirds. In each case, is it greater than, less than, or equal to an octave, and by how much in cents?

Just 3rd is \(\frac{3}{2}\).

\[
\text{Since } \frac{125}{64} < \frac{129}{64} \Rightarrow \text{ it is less than an octave}
\]

In cents the difference is
\[
12\log \left( \frac{129}{125} \right) \approx 41 \text{ cents}.
\]

Pythagorean 3rd is \(\left(\frac{3}{2}\right)^4/4 = \frac{3^4}{2^6} = \frac{81}{64}\)

\[\text{ HAVE A GOOD SUMMER!} \]

3 Pyth 3rds is \(\left(\frac{81}{64}\right)^3/2 \approx 2.63 > 2 \text{ so greater than an octave.} \]

In cents the difference is
\[
12\log \left[ \left(\frac{81}{64}\right)^3/2 \right] \approx 13 \text{ cents}.
\]