Solutions of Practice Exam.

Problem 1
(Recall the definition of the derivative.) \( f(x) \) has the derivative at \( p \) means that \( f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h} \) also exists by the assumption. Therefore

\[
(f + g)'(p) = \lim_{h \to 0} \frac{(f + g)(p + h) - (f + g)(p)}{h} = \lim_{h \to 0} \frac{f(p + h) - f(p) + g(p + h) - g(p)}{h} = \left( \lim_{h \to 0} \frac{f(p + h) - f(p)}{h} \right) + \left( \lim_{h \to 0} \frac{g(p + h) - g(p)}{h} \right) = f'(p) + g'(p)
\]
also exists.

Problem 2
By the first fundamental theorem of calculus, \( f'(x) = \frac{1}{1+x^2} \). By solving \( f'(2) = \frac{1}{a+2^2} = \frac{1}{11} \), we obtain \( a = 3 \).

Problem 3
(a) (Note that \( f(x) \) is a continuous function, hence we actually can apply Bolzano’s theorem.) Since \( f(0) = -2 < 0 \) and \( f(1) = \sin(1) > 0 \), by the theorem, there exists at least one real root in \((0, 1)\).
(b) \( f(x) \) is also a differentiable function, hence we can apply Roll’s theorem.) If \( f(x) \) has two distinct real roots \( \alpha \) and \( \beta \) \((\alpha < \beta)\) in \((0, 1)\), by Roll’s theorem, there must be at least one \( c \in (\alpha, \beta) \) such that \( f'(c) = 0 \). However, \( f'(x) = 4x^3 + 1 + \cos x > 0 \) over \((0, 1)\) (note that \( \cos x > 0 \) over \((0, 1)\)). Contradiction.

Problem 4
( Check the definitions of the inner product, cross product, and projection.)
(a) \((1 \cdot 3) + (-1 \cdot 2) + (2 \cdot 1) = 3\).
(b)
\[
\begin{pmatrix}
(-1 \cdot 1) - (2 \cdot 2) \\
2 \cdot 3 - (1 \cdot 1) \\
1 \cdot 2 - (-1) \cdot 3)
\end{pmatrix} = \begin{pmatrix}
-5 \\
5 \\
5
\end{pmatrix}
\]
(c) \( \|B\|^2 = B \cdot B = 3^2 + 2^2 + 1^2 = 14 \). Hence the projection is given by
\[
\left( \frac{A \cdot B}{\|B\|^2} \right) B = \frac{3}{14} \begin{pmatrix}
3 \\
2 \\
1
\end{pmatrix}.
\]
Problem 5
By the assumption,
\[ 0 = (2X + Y) \cdot Z = \left( \begin{array}{c} 4 \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 3 \\ \alpha \end{array} \right) = 12 + \alpha. \]
Hence \( \alpha = -12. \)

Problem 6
(1) Since \( V \) and \( W \) are subspaces, both of them contains the zero vector 0. Hence \( 0 \in V \cap W. \)
(2) Take any two vectors \( x, y \in V \cap W. \) Since \( V \) is a subspace, \( x + y \in V. \) Similarly \( x + y \in W. \) Hence \( x + y \in V \cap W. \)
(3) Take any vector \( x \in V \cap W \) and an arbitrary real number \( c. \) Since \( V \) is a subspace, \( cx \in V. \) Similarly \( cx \in W. \) Hence \( cx \in V \cap W. \)

Problem 7
\[ f'(x) = 3x^2 - 6x = 3x(x - 2). \] Hence it suffices to compare the values of \( f(x) \) at \( x = 0, 2 \) and at the other boundary point \( x = -\frac{1}{2} \) of the given interval. Since \( f(-\frac{1}{2}) = \frac{9}{8}, f(0) = 2, f(2) = -2, \) the maximum is 2 and the minimum is -2.

Problem 8
\[ f(0) = 0 \text{ and } f(\pi) = \pi^2 - 2\pi. \] Since \( f(x) \) is continuous, by the intermediate-value theorem, \( f(x) \) can take any value in \((0, \pi^2 - 2\pi)\) over the given interval. Since \( 0 < 1 < \pi^2 - 2\pi, \) there exists at least one required \( a. \)

Problem 9
(a) If \( a, b, c \in \mathbb{R} \) satisfies the equation \( aA + bB + cC = 0, \) it implies
\[ a + 2b - 3c = 0 \]
\[ 3a + 0 + 2c = 0 \]
\[ a - 2b - 3c = 0 \]
Their only solution is \( a = b = c = 0, \) hence \( \{A, B, C\} \) is consisted by three linearly independent vectors in \( \mathbb{R}^3. \) Thus it is a basis. Moreover, easily we can check that \( A \cdot B = B \cdot C = C \cdot A = 0 \) by the direct computation. Hence \( A, B, C \) are orthogonal to each other. (b)(Recall that for a given vector \( V, V' := \frac{V}{||V||} \) has the same direction with \( V \) and \( ||V'|| = 1). \) Since \( ||A|| = \sqrt{11}, ||B|| = 2\sqrt{2}, ||C|| = \sqrt{22}, \)
\[ \left\{ \frac{1}{\sqrt{11}}A, \frac{1}{2\sqrt{2}}B, \frac{1}{\sqrt{22}}C \right\} \]
is consisted by unit vectors. Since each of these vectors has the same direction with \( A, B, \) and \( C \) respectively, it is an orthonormal basis.