Yet another point of view on affine geometry

Symposium in honor of Alan Weinstein
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Affine connections

Definition:

\[ \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y \]

\[ \nabla_{fX} Y = f \nabla_X Y \]

\[ \nabla_X fY = Xf \, Y + f \nabla_X Y \] (Leibniz rule).

\[ \nabla_{\dot{\gamma}(t)} V \] derivative of a vector field along a path.
Definition: A geodesic is a path $\gamma$ of constant velocity, i.e.

$$\nabla \dot{\gamma}(t) \dot{\gamma} = 0.$$ 

Existence and uniqueness: $X \in T_x M$ determines a unique (maximal) geodesic $t \mapsto \exp_x(tX)$ tangent to $X$.

$\rightsquigarrow$ The geodesic symmetry around $x$ is $s_x(\exp_x(tX)) = \exp_x(-tX)$
Their automorphisms

**Definition:** A 2-jet $j^2_x f$ ($f : x \in U \subset M \xrightarrow{\sim} V \subset M$) is **affine** if

$$f^* x \left( \nabla_X Y \right)_x = \nabla_{f^* x X_x} f^* Y.$$ 

A diffeomorphism $f : M \to M$ is affine if $j^2 f$ is affine everywhere.
Their tensors

Torsion

\[ T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \]

Curvature

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z. \]

Their covariant derivatives

\[ (\nabla_X T)(Y, Z) = \nabla_X (T(Y, Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z). \]

\[ \vdots \]

\[ \vdots \]

\[ \vdots \]

No other tensor! ... Why?
Canonical affine connection of a symmetric space

On a symmetric space \((M, s = (s_x)_{x \in M})\)

- \(s_x^2 = \text{id}\)
- \(s_x(x) = x\)
- \((s_x)_* = -I\)
- \(s_x \circ s_y \circ s_x = s_{s_x(y)}\),

the formula

\[
\left( \nabla^s_X Y \right)_x = \frac{1}{2} \left[ X, Y + (s_x)_* Y \right]_x
\]

defines the affine connection for which the \(s_x\) are affine maps. It is torsionless!
Affine connections and symmetry jets
Affine extensions
Holonomy and torsion, curvature, ...

Observations

- $\nabla^s_x$ depends only on $j_x^2 s_x$,
- the $s_x$’s are the geodesic symmetries,

For any affine torsionless connection $\nabla$,

$$
\left( \nabla_X Y \right)_x = \frac{1}{2} \left[ X, Y + (s_x)_* Y \right]_x,
$$

provided $s_x = \text{geodesic symmetry at } x$.

**Proposition**: torsionless affine connections $\iff$ symmetry jets $s$

$$
x \mapsto s(x) = j_x^2 s_x \quad \text{with} \quad j_x^1 s_x = -l_x.
$$

**Remark**: Same with torsion if allow non-holonomic jets

$$
j_x^1 j_x^1 f. \quad \text{with} \quad j_x^1 f. : x' \mapsto j_x^1 f_x',
$$
General linear groupoid

\[ GL(TM) = B^{(1)}(M) = \left\{ \xi = j_{x}^{1}f : T_{x}M \xrightarrow{\sim} T_{f(x)}M \right\}. \]

\[ GL(TM) \times_{\alpha,p} TM \rightarrow TM : (\xi, X_{\alpha}(\xi)) \mapsto \xi \cdot X_{\alpha}(\xi). \]
A $(1, 1)$-jet $j^1_x j^1 f$ is the 1-jet at $x$ of the local bisection

$$x' \in U \subset M \mapsto j^1_x f_{x'} \in \mathcal{B}^{(1)}(M).$$

$(1, 1)$-jets $\longleftrightarrow$ $n$-planes on $\mathcal{B}^{(1)}(M)$ transverse to $\alpha, \beta$:

$$j^1_x j^1 f \longleftrightarrow \left( j^1 f \right)_x (T_x M) \subset T_{j^1_x f} \mathcal{B}^{(1)}(M).$$
Symmetry jets and distributions

Symmetry jet $\leftrightarrow$ distributions $\mathcal{D}_{-l}$ along $-l$ on $\mathcal{B}^{(1)}(M)$.

$\alpha$ $\beta$

$\mathcal{D}_{-l_x} \mathcal{D}_{-l_{x'}}$

$\mathcal{B}^{(1)}(M)$

$M$

$-l$

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Well-known:

$$\nabla_{\xi} : T_xM \simarrow T_yM$$

$$\implies S(\xi) = j^2_x f$$ affine such that $$j^1_x f = \xi$$.

In fact there exists a unique groupoid morphism

$$S : \mathcal{B}^{(1)}(M) \rightarrow \mathcal{B}^{(2)}(M)$$

$$S(-l_x) = s(x).$$

Same result for connexions with torsion $$\mathcal{B}^{(2)}(M) \leftarrow \mathcal{B}^{(1,1)}(M)$$
the groupoid of 1-jets of local bisections of $$\mathcal{B}^{(1)}(M)$$ (tangent $$n$$-planes transverse to $$\alpha$$ and $$\beta$$).
$S \leftrightarrow \mathcal{D}^S$ a distribution on $\mathcal{B}^{(1)}(M)$ (leaves are affine maps).
Observation: A \((1, 1)\)-jet that is a 2-jet is invariant under \(\kappa\), where \(\kappa\) = canonical involution on \(\mathcal{B}^{(1,1)}\) that permutes the order of derivations.

Proposition:

\[
S(\xi) \cdot \mathcal{X} - \kappa(S(\xi)) \cdot \mathcal{X} \overset{(\ast)}{=} \xi \left( T(X_x, Y_x) \right) - T(\xi(X_x), \xi(Y_x)).
\]

\(\mathcal{X} = Y_x^* X_x \in T^2 M\) on which \(\mathcal{B}^{(1,1)}(M)\) acts.

\(S(\xi)\) holonomic \(\text{iff}\) \(\xi\) preserves \(T\).
If $\xi \in \mathcal{B}^{(1)}(M)$, let $a_{\xi}$ = local bisection of $\mathcal{B}^{(1)}$ tangent to $\mathcal{D}_{\xi}$.

$$\tilde{S} \circ a_{\xi} \text{ a local bisection of } \mathcal{B}^{(1,1)}(M).$$

$$S^{(3)}(\xi) = j_{\mathcal{D}_{\xi}}^{(1)}(S \circ a_{\xi}) \text{ is an affine } (1,1,1)\text{-jet}.$$ 

$$\xi \left( \nabla_{\nabla_{X} X} \nabla_{Y} Z \right) = \nabla_{\nabla_{X} X} a_{\xi} \left( \nabla_{Y} Z \right) = \nabla_{\nabla_{X} X} a_{\xi} Y a_{\xi}(.) Z.$$ 

In fact $S^{(3)}(\xi)$ corresponds to the $n$-plane tangent to $\mathcal{B}^{(1,1)}(M)$:

$$S_{\mathcal{D}_{\xi}}^{\ast}(\mathcal{D}_{\xi}).$$
Observation: \((1,1,1)\)-jet holonomic \(\iff\) \(\kappa\) and \(\kappa^*\)-invariant \((\kappa\) permutes order of last two derivatives, \(\kappa^*\) permutes order of first two derivatives).

Differentiating \((\ast)\) yields

\[
S^{(3)}(\xi) \cdot \mathcal{X} - \kappa^*(S^{(3)}(\xi)) \cdot \mathcal{X} = \xi \left( (\nabla_{Z_x} T)(Y_x, X_x) \right) - \left( \nabla_{\xi Z_z} T \right)(\xi Y_x, \xi X_x),
\]

where \(\mathcal{X} = Z_{**y_x} Y_{*x} X_x \in T^3M\) on which \(B^{(1,1,1)}(M)\) acts.
One can prove that $\kappa$-invariance depends on the curvature:

$$S^{(3)}(\xi) \cdot \mathcal{X} - \kappa(S^{(3)}(\xi)) \cdot \mathcal{X} = \xi \left( R \nabla (X_x, Y_x) Z_x \right) - R \nabla (\xi X_x, \xi Y_x) \xi Z_x,$$

where, again, $\mathcal{X} = Z_{*y_x} Y_{*x} X_x$.

$S^{(3)}(\xi)$ holonomic if and only if $\xi$ preserves $\nabla T, R$. 
Tensor Subgroupoids

**Notation**: $P_1, \ldots, P_k = $ covariant tensors on $M$, values in $TM$,

$$\mathcal{B}(P_1, \ldots, P_k) = \left\{ \xi \in \mathcal{B}^{(1)}(M) \mid \xi P_i(\cdot) = P_i(\xi \cdot) \quad i = 1, \ldots, k \right\}.$$

If $s$ is a symmetry jet $\sim \mathcal{B}(T), \mathcal{B}(R), \mathcal{B}(T, \nabla T, R), \ldots$. 
Notation: $\mathcal{D}$ is a distribution on $W$,

$$\text{Int(}\mathcal{D}\text{)} = \big\{ w \in W \big| [\mathcal{D}, \mathcal{D}]_w \subset \mathcal{D}_w \big\}.$$ 

Proposition: $\text{Int(}\mathcal{D}^S\text{)} = \mathcal{B}(T, R)$.

$$\xi \in \text{Int(}\mathcal{D}^S\text{)} \implies \begin{cases} S^{(3)} \text{ is } \kappa \text{ -- invariant} \\ \xi \text{ preserves } R. \end{cases}$$
Next order affine extension

Affine jets exist at all orders:

\[ S^{(4)}(\xi) = j^1_x (S^{(3)} \circ a_\xi) \]

which corresponds to the \( n \)-plane tangent to \( B^{(1,1,1)}(M) \)

\[ S^{(3)}_{*\xi}(D_\xi). \]

Holonomy of \( S^{(4)}(\xi) \) depends on invariance under

\[ \kappa_{**} \leftrightarrow \nabla \nabla T \]
\[ \kappa_* \leftrightarrow \nabla R \]
\[ \kappa \leftrightarrow ?? \]
$S^{(4)}(\xi)$ is automatically $\kappa$-invariant

$S^{(4)}(\xi)$ is $\kappa$-invariant provided $S(\xi) \in \text{Int}(S_\ast \mathcal{D}^5)$.

\[
S(\xi) \in \text{Int}(S_\ast \mathcal{D}^5) \iff [S_\ast \mathcal{D}^5, S_\ast \mathcal{D}^5]_S(\xi) \subset S_{\ast \xi}(\mathcal{D}^5 \xi)
\]

\[
\iff S_{\ast \xi} [\mathcal{D}^5, \mathcal{D}^5]_\xi \subset S_{\ast \xi}(\mathcal{D}^5 \xi)
\]

\[
\iff [\mathcal{D}^5, \mathcal{D}^5]_\xi \subset \mathcal{D}^5 \xi
\]

\[
\iff \xi \in \text{Int}(\mathcal{D}^5)
\]

So $?? = 0$!!!
Happy Birthday Alan!

Thank you for sharing your love for mathematics with us.