Examples of Poisson brackets

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Definition of Poisson algebras

Definition. A Poisson algebra over a field $\mathbf{k}$ is a commutative $\mathbf{k}$-algebra A together with a $\mathbf{k}$-bilinear skew-symmetric map $\{,\} : A \times A \longrightarrow A$, called the *Poisson bracket*, such that for all $a, b, c \in A$,

Leibniz rule :

$$\{a, bc\} = b\{a, c\} + c\{a, b\},$$

Jacobi identity :

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0.$$

A Poisson space is a space X with a Poisson algebra structure on its algebra of functions.

Question

Why are Poisson structures interesting?

Answer

"They become interesting if you study them long enough"

—Alen Weinstein, 1985

Examples?

In local coordinates $(x_1, x_2, \ldots, x_n)$, $\{\ ,\ \}$ is determined by the $\binom{n}{2}$ functions

$$\{x_i, x_j\}, \quad 1 \leq i < j \leq n,$$

which must satisfy $\binom{n}{3}$ non-linear PDEs.

Poisson structures are hard to make up.

Standard Examples

0) $\{x_i, x_j\}$ is cnstant for all $1 \leq i < j \leq n$;

1) $\{x_i, x_j\}$ is linear for all $1 \leq i < j \leq n$;

2) $\{x_i, x_j\} = c_{ij} x_i x_j$ for constants c_{ij}.

3) X or X/G, where X is symplectic, G acting on X by symplectomorphisms

Examples in Nature?

Quantum spaces give rise to Poisson structures as semi-classical limits;

Quantum groups give rise to Poisson structures on (Lie) groups and related spaces.

Goal of talk: To explain some of these Poisson spaces.

Basic constructions

A Lie bialgebra is a Lie algebra $\mathfrak{g}$ together with a compatible Lie algebra structure on $\mathfrak{g}^*$.

Constructions:

- Lie bialgebras $\Longrightarrow$ Poisson Lie groups: pairs (G, π), where G is a Lie group and π a multiplicative Poisson structure on G;
- Quotiens.

Three classes

- Real semi-simple Lie groups $\Longrightarrow$ real analytic Poisson structures;
- Complex semi-simple Lie groups $\Longrightarrow$ holomorphic Poisson structures;
- Reductive algebraic groups $\Longrightarrow$ algebraic Poisson structures, even over positive characteristics.

Common features

- "Natural Darboux type" coordinates;
- Finitely many T-leaves for some torus T.

Real semi-simple Lie groups

G_0, real semisimple Lie group;
$G = (G_0)_{\mathbb{C}}$, complexification of G_0;
$U \subset G$, maximal compact subgroup of G;
$K_0 = U \cap G_0$, maximal compact subgroup of G_0;
$\mathcal{B}$, flag variety of G.

Example: $G_0 = SL(n, \mathbb{R})$, $G = SL(n, \mathbb{C})$, $U = SU(n)$, $K_0 = SO(n)$.

Spaces with natural real analytic Poisson structures:

- G_0: one standard multiplicative structure for each open G_0-orbit in $\mathcal{B}$;
- non-compact symmetric space G_0/K_0;
- compact symmetric space U/K_0;
- flag manifold $\mathcal{B}$.

Poisson structures on $\mathcal{B}$ coming from G_0

- A choice of a Borel subgroup B in an open G_0-orbit in $\mathcal{B}$ defines a Poisson structure on $\mathcal{B}$;
- T_0-orbits of symplectic leaves are connected comopnents of $(G_0-\text{orbits}) \cap (B-\text{orbits})$;
- There are finitely many G_0-orbits and B-orbits in $\mathcal{B}$; B-orbits parametrized by the Weyl group.
- Matsuki correspondence between G_0-orbits and $K = (K_0)_{\mathbb{C}}$-orbits in $\mathcal{B}$;

- The geometry and combinatorics of K-orbit closures are essential in representations of G_0, eg. Vogan-Kazdan-Lusztig polynomials;
- Computations of T_0-leaves can be done using Atlas of Lie groups.
- When $G_0 = U$, the Poisson structure on $\mathcal{B}$ is closely related to Kostant's harmonic forms and Schubert calculus on $\mathcal{B}$.

Reductive algebraic groups

G, connected reductive algebraic group over $\mathbf{k} = \overline{\mathbf{k}}$;

Spaces carrying natural algebraic Poisson structures:

- G and the wonderful compactification $\overline{G}$ of G;
- (Twisted) conjugacy classes of G;
- Grothendieck simultaneous resolution of G; Unipotent varieties; Steinberg fibers;
- Flag varietties G/B, G/P, and products
$$G/B \times G/B \times \cdots \times G/B;$$
- Bott-Samelson varieties.

Examples coming from Bott-Samelson varieties

Let S be the set of simple reflections in Weyl group W, $e \in W$ the identity element.

- A sequence $\mathbf{u} = (s_1, s_2, \ldots, s_n)$ in S defines a Bott-Samelson variety of dim $= n$:

$$Z_{\mathbf{u}} = P_{s_1} \times_B P_{s_2} \times_B \cdots \times_B P_{s_n} / B.$$

- A subexpression γ of $\mathbf{u}$, i.e.,

$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n),$$

where $\gamma_j \in \{s_j, e\}$, defines an open subset $\mathcal{O}^\gamma \cong \mathbb{C}^n$ with coordinates $(x_1, x_2, \ldots, x_n)$;

Facts:

- The n-fold product of the standard multiplicative Poisson structure on G descends to a well-defined Poisosn structure $\Pi_{\mathbf{u}}$ on $Z_{\mathbf{u}}$.
- In the coordinates $(x_1, x_2, \ldots, x_n)$ on $\mathcal{O}^\gamma$,

  $$\{x_i, x_j\} \in \mathbb{Z}[x_1, x_2, \ldots, x_n], \quad \forall\, 1 \leq i < j \leq n,$$

  so have a Poisson bracket on $\mathbf{k}[x_1, x_2, \ldots, x_n]$ for any field $\mathbf{k}$;
- Finitely many T-leaves!
- Computer program by Balazs Elek.

Example 3. $G = G_2$, $\mathbf{s} = (s_1, s_2, s_1, s_1, s_2)$ and $\gamma = (s_1, s_2, e, s_1, e)$,

$$
\begin{aligned}
\{x_1, x_2\} &= -3x_1x_2\\
\{x_1, x_3\} &= 2x_2x_3^2 + x_1x_3\\
\{x_1, x_4\} &= -4x_2x_3x_4 - x_1x_4 - 2x_2\\
\{x_1, x_5\} &= 6x_3x_4^3x_5^2 + 6x_4^2x_5^2 + 6x_2x_3x_5\\
\{x_2, x_3\} &= 3x_2x_3\\
\{x_2, x_4\} &= -3x_2x_4\\
\{x_2, x_5\} &= 6x_4^3x_5^2 + 3x_2x_5\\
\{x_3, x_4\} &= -2x_3x_4\\
\{x_3, x_5\} &= 3x_3x_5\\
\{x_4, x_5\} &= 3x_4x_5.
\end{aligned}
$$

Let's check the Jacobi identity:

$$\{\{x_1, x_2\}, x_5\} = -18x_2x_3x_4^3x_5^2 - 18x_1x_4^3x_5^2 - 18x_2x_4^2x_5^2 - 18x_2^2x_3x_5 - 9x_1x_2x_5,$$

$$\{\{x_2, x_5\}, x_1\} = -72x_3x_4^6x_5^3 - 72x_4^5x_5^3 - 18x_2x_3x_4^3x_5^2 + 18x_1x_4^3x_5^2 + 18x_2x_4^2x_5^2 - 18x_2^2x_3x_5 + 9x_1x_2x_5,$$

$$\{\{x_5, x_1\}, x_2\} = 72x_3x_4^6x_5^3 + 72x_4^5x_5^3 + 36x_2x_3x_4^3x_5^2 + 36x_2^2x_3x_5$$

$$\{\{x_1, x_2\}, x_5\} + \{\{x_2, x_5\}, x_1\} + \{\{x_5, x_1\}, x_2\} = 0.$$

Another example. $G = G_2$,

$\mathbf{w} = (s_1, s_2, s_1, s_1, s_2, s_1, s_1, s_1, s_2)$,
$\gamma = (s_1, s_2, e, s_1, s_2, e, s_1, s_1, e)$,

$$\{x_1, x_2\} = -3x_1x_2$$
$$\{x_1, x_3\} = 2x_2x_3^2 + x_1x_3$$
$$\{x_1, x_4\} = -4x_2x_3x_4 - x_1x_4 - 2x_2$$
$$\{x_1, x_5\} = -6x_3x_4^3 - 6x_2x_3x_5 - 6x_4^2$$
$$\{x_1, x_6\} = 6x_3x_4^2x_6^2 + 2x_2x_3x_6 + 4x_4x_6^2 - x_1x_6$$
$$\{x_1, x_7\} = -12x_3x_4^2x_6x_7 - 2x_2x_3x_7 - 6x_3x_4^2 - 8x_4x_6x_7 + x_1x_7 - 4x_4$$
$$\{x_1, x_8\} = 12x_3x_4^2x_6x_8 + 2x_2x_3x_8 + 8x_4x_6x_8 - x_1x_8$$

$$
\begin{aligned}
\{x_1, x_9\} = & -6x_3x_5x_6^3x_7^3x_8^3x_9^2 + 18x_3x_4x_6^2x_7^3x_8^3x_9^2 \\
& + 18x_3x_5x_6^3x_7^2x_8^2x_9^2 - 18x_3x_5x_6^2x_7^2x_8^3x_9^2 \\
& - 54x_3x_4x_6^2x_7^2x_8^2x_9^2 + 36x_3x_4x_6x_7^2x_8^3x_9^2 \\
& + 6x_6^2x_7^3x_8^3x_9^2 - 18x_3x_5x_6^3x_7x_8x_9^2 \\
& + 36x_3x_5x_6^2x_7x_8^2x_9^2 - 18x_3x_5x_6x_7x_8^3x_9^2 \\
& + 54x_3x_4x_6^2x_7x_8x_9^2 - 72x_3x_4x_6x_7x_8^2x_9^2 \\
& + 18x_3x_4x_7x_8^3x_9^2 - 18x_6^2x_7^2x_8^2x_9^2 + 12x_6x_7^2x_8^3x_9^2 \\
& + 6x_3x_5x_6^3x_9^2 - 18x_3x_5x_6^2x_8x_9^2 + 18x_3x_5x_6x_8^2x_9^2 \\
& - 6x_3x_5x_8^3x_9^2 - 18x_3x_4x_6^2x_9^2 + 36x_3x_4x_6x_8x_9^2
\end{aligned}
$$

$$
\begin{aligned}
&- 18x_3x_4x_8^2x_9^2 + 18x_6^2x_7x_8x_9^2 - 24x_6x_7x_8^2x_9^2 \\
&+ 6x_7x_8^3x_9^2 - 18x_3x_4^2x_6x_9 - 6x_6^2x_9^2 \\
&+ 12x_6x_8x_9^2 - 6x_8^2x_9^2 - 6x_2x_3x_9 - 12x_4x_6x_9
\end{aligned}
$$

$$
\begin{aligned}
\{x_2, x_3\} &= 3x_2x_3 \\
\{x_2, x_4\} &= -3x_2x_4 \\
\{x_2, x_5\} &= -6x_4^3 - 3x_2x_5 \\
\{x_2, x_6\} &= 6x_4^2x_6^2 \\
\{x_2, x_7\} &= -12x_4^2x_6x_7 - 6x_4^2 \\
\{x_2, x_8\} &= 12x_4^2x_6x_8.
\end{aligned}
$$

$$
\begin{aligned}
\{x_2, x_9\} = &-6x_5x_6^3x_7^3x_8^3x_9^2 + 18x_4x_6^2x_7^3x_8^3x_9^2 \\
&+ 18x_5x_6^3x_7^2x_8^2x_9^2 - 18x_5x_6^2x_7^2x_8^3x_9^2 \\
&- 54x_4x_6^2x_7^2x_8^2x_9^2 + 36x_4x_6x_7^2x_8^3x_9^2 \\
&- 18x_5x_6^3x_7x_8x_9^2 + 36x_5x_6^2x_7x_8^2x_9^2 \\
&- 18x_5x_6x_7x_8^3x_9^2 + 54x_4x_6^2x_7x_8x_9^2 \\
&- 72x_4x_6x_7x_8^2x_9^2 + 18x_4x_7x_8^3x_9^2 + 6x_5x_6^3x_9^2 \\
&- 18x_5x_6^2x_8x_9^2 + 18x_5x_6x_8^2x_9^2 - 6x_5x_8^3x_9^2 \\
&- 18x_4x_6^2x_9^2 + 36x_4x_6x_8x_9^2 - 18x_4x_8^2x_9^2 \\
&- 18x_4^2x_6x_9 - 3x_2x_9,
\end{aligned}
$$

$$\{x_3, x_4\} = -2x_3x_4$$
$$\{x_3, x_5\} = -3x_3x_5$$
$$\{x_3, x_6\} = x_3x_6$$
$$\{x_3, x_7\} = -x_3x_7$$
$$\{x_3, x_8\} = x_3x_8$$
$$\{x_3, x_9\} = -3x_3x_9$$
$$\{x_4, x_5\} = -3x_4x_5$$
$$\{x_4, x_6\} = 2x_5x_6^2 + x_4x_6$$
$$\{x_4, x_7\} = -4x_5x_6x_7 - x_4x_7 - 2x_5$$
$$\{x_4, x_8\} = 4x_5x_6x_8 + x_4x_8$$

$$
\begin{aligned}
\{x_4, x_9\} &= 6x_6x_7^3x_8^3x_9^2 - 18x_6x_7^2x_8^2x_9^2 \\
&\quad + 6x_7^2x_8^3x_9^2 + 18x_6x_7x_8x_9^2 - 12x_7x_8^2x_9^2 \\
&\quad - 6x_5x_6x_9 - 6x_6x_9^2 + 6x_8x_9^2 - 3x_4x_9 \\
\{x_5, x_6\} &= 3x_5x_6 \\
\{x_5, x_7\} &= -3x_5x_7 \\
\{x_5, x_8\} &= 3x_5x_8 \\
\{x_5, x_9\} &= 6x_7^3x_8^3x_9^2 - 18x_7^2x_8^2x_9^2 + 18x_7x_8x_9^2 \\
&\quad - 6x_5x_9 - 6x_9^2 \\
\{x_6, x_7\} &= -2x_6x_7, \quad \{x_6, x_8\} = 2x_6x_8 \\
\{x_6, x_9\} &= -3x_6x_9, \quad \{x_7, x_8\} = 2x_7x_8 - 2 \\
\{x_7, x_9\} &= -3x_7x_9, \quad \{x_8, x_9\} = 3x_8x_9
\end{aligned}
$$

$$
\begin{aligned}
\{x_1, \{x_5, x_9\}\}
&= -72x_3x_5x_6^3x_7^6x_8^6x_9^3 + 216x_3x_4x_6^2x_7^6x_8^6x_9^3 \\
&+ 432x_3x_5x_6^3x_7^5x_8^5x_9^3 - 216x_3x_5x_6^2x_7^5x_8^6x_9^3 \\
&- 1296x_3x_4x_6^2x_7^5x_8^5x_9^3 + 432x_3x_4x_6x_7^5x_8^6x_9^3 \\
&+ 72x_6^2x_7^6x_8^6x_9^3 - 1080x_3x_5x_6^3x_7^4x_8^4x_9^3 \\
&+ 1080x_3x_5x_6^2x_7^4x_8^5x_9^3 - 216x_3x_5x_6x_7^4x_8^6x_9^3 \\
&+ 3240x_3x_4x_6^2x_7^4x_8^4x_9^3 - 2160x_3x_4x_6x_7^4x_8^5x_9^3 \\
&+ 216x_3x_4x_7^4x_8^6x_9^3 - 432x_6^2x_7^5x_8^5x_9^3 \\
&+ 144x_6x_7^5x_8^6x_9^3 + 36x_3x_5^2x_6^3x_7^3x_8^3x_9^2 \\
&+ 1440x_3x_5x_6^3x_7^3x_8^3x_9^3 - 2160x_3x_5x_6^2x_7^3x_8^4x_9^3
\end{aligned}
$$

$$
\begin{aligned}
&+ 864x_3x_5x_6x_7^3x_8^5x_9^3 - 72x_3x_5x_7^3x_8^6x_9^3 \\
&- 108x_3x_4x_5x_6^2x_7^3x_8^3x_9^2 - 4320x_3x_4x_6^2x_7^3x_8^3x_9^3 \\
&+ 4320x_3x_4x_6x_7^3x_8^4x_9^3 - 864x_3x_4x_7^3x_8^5x_9^3 \\
&+ 1080x_6^2x_7^4x_8^4x_9^3 - 720x_6x_7^4x_8^5x_9^3 + 72x_7^4x_8^6x_9^3 \\
&- 216x_3x_4^2x_6x_7^3x_8^3x_9^2 - 108x_3x_5^2x_6^3x_7^2x_8^2x_9^2 \\
&+ 108x_3x_5^2x_6^2x_7^2x_8^3x_9^2 - 1080x_3x_5x_6^3x_7^2x_8^2x_9^3 \\
&+ 2160x_3x_5x_6^2x_7^2x_8^3x_9^3 - 1296x_3x_5x_6x_7^2x_8^4x_9^3 \\
&+ 216x_3x_5x_7^2x_8^5x_9^3 + 324x_3x_4x_5x_6^2x_7^2x_8^2x_9^2 \\
&- 216x_3x_4x_5x_6x_7^2x_8^3x_9^2 + 3240x_3x_4x_6^2x_7^2x_8^2x_9^3
\end{aligned}
$$

$$
\begin{aligned}
&- 4320x_3x_4x_6x_7^2x_8^3x_9^3 + 1296x_3x_4x_7^2x_8^4x_9^3 \\
&- 36x_5x_6^2x_7^3x_8^3x_9^2 - 1440x_6^2x_7^3x_8^3x_9^3 \\
&+ 1440x_6x_7^3x_8^4x_9^3 - 288x_7^3x_8^5x_9^3 \\
&- 72x_2x_3x_7^3x_8^3x_9^2 + 648x_3x_4^2x_6x_7^2x_8^2x_9^2 \\
&- 108x_3x_4^2x_7^2x_8^3x_9^2 + 108x_3x_5^2x_6^3x_7x_8x_9^2 \\
&- 216x_3x_5^2x_6^2x_7x_8^2x_9^2 + 108x_3x_5^2x_6x_7x_8^3x_9^2 \\
&+ 432x_3x_5x_6^3x_7x_8x_9^3 - 1080x_3x_5x_6^2x_7x_8^2x_9^3 \\
&+ 864x_3x_5x_6x_7x_8^3x_9^3 - 216x_3x_5x_7x_8^4x_9^3 \\
&- 144x_4x_6x_7^3x_8^3x_9^2 - 324x_3x_4x_5x_6^2x_7x_8x_9^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 432x_3x_4x_5x_6x_7x_8^2x_9^2 - 108x_3x_4x_5x_7x_8^3x_9^2 \\
&- 1296x_3x_4x_6^2x_7x_8x_9^3 + 2160x_3x_4x_6x_7x_8^2x_9^3 \\
&- 864x_3x_4x_7x_8^3x_9^3 + 108x_5x_6^2x_7^2x_8^2x_9^2 - 72x_5x_6x_7^2x_8^3x_9^2 \\
&+ 1080x_6^2x_7^2x_8^2x_9^3 - 1440x_6x_7^2x_8^3x_9^3 \\
&+ 432x_7^2x_8^4x_9^3 + 216x_2x_3x_7^2x_8^2x_9^2 - 648x_3x_4^2x_6x_7x_8x_9^2 \\
&+ 216x_3x_4^2x_7x_8^2x_9^2 - 36x_3x_5^2x_6^3x_9^2 + 108x_3x_5^2x_6^2x_8x_9^2 \\
&- 108x_3x_5^2x_6x_8^2x_9^2 + 36x_3x_5^2x_8^3x_9^2 - 72x_3x_5x_6^3x_9^3 \\
&+ 216x_3x_5x_6^2x_8x_9^3 - 216x_3x_5x_6x_8^2x_9^3 + 72x_3x_5x_8^3x_9^3 \\
&+ 432x_4x_6x_7^2x_8^2x_9^2 - 72x_4x_7^2x_8^3x_9^2 + 108x_3x_4x_5x_6^2x_9^2
\end{aligned}
$$

$$
\begin{aligned}
&- 216x_3x_4x_5x_6x_8x_9^2 + 108x_3x_4x_5x_8^2x_9^2 + 216x_3x_4x_6^2x_9^3 \\
&- 432x_3x_4x_6x_8x_9^3 + 216x_3x_4x_8^2x_9^3 - 108x_5x_6^2x_7x_8x_9^2 \\
&+ 144x_5x_6x_7x_8^2x_9^2 - 36x_5x_7x_8^3x_9^2 - 432x_6^2x_7x_8x_9^3 \\
&+ 720x_6x_7x_8^2x_9^3 - 288x_7x_8^3x_9^3 - 216x_2x_3x_7x_8x_9^2 \\
&+ 108x_3x_4^2x_5x_6x_9 + 216x_3x_4^2x_6x_9^2 - 108x_3x_4^2x_8x_9^2 \\
&- 432x_4x_6x_7x_8x_9^2 + 144x_4x_7x_8^2x_9^2 + 36x_3x_4^3x_9 \\
&+ 36x_5x_6^2x_9^2 - 72x_5x_6x_8x_9^2 + 36x_5x_8^2x_9^2 + 72x_6^2x_9^3 \\
&- 144x_6x_8x_9^3 + 72x_8^2x_9^3 + 72x_2x_3x_5x_9 + 72x_2x_3x_9^2 \\
&+ 72x_4x_5x_6x_9 + 144x_4x_6x_9^2 - 72x_4x_8x_9^2 + 36x_4^2x_9
\end{aligned}
$$

$$
\begin{aligned}
&- 324x_3x_4^2x_6x_7^2x_8^2x_9^2 - 432x_3x_5x_6^3x_7x_8x_9^3 \\
&+ 1080x_3x_5x_6^2x_7x_8^2x_9^3 - 864x_3x_5x_6x_7x_8^3x_9^3 \\
&+ 216x_3x_5x_7x_8^4x_9^3 + 72x_4x_6x_7^3x_8^3x_9^2 \\
&+ 1296x_3x_4x_6^2x_7x_8x_9^3 - 2160x_3x_4x_6x_7x_8^2x_9^3 \\
&+ 864x_3x_4x_7x_8^3x_9^3 - 108x_5x_6^2x_7^2x_8^2x_9^2 \\
&+ 72x_5x_6x_7^2x_8^3x_9^2 - 1080x_6^2x_7^2x_8^2x_9^3 \\
&+ 1440x_6x_7^2x_8^3x_9^3 - 432x_7^2x_8^4x_9^3 \\
&- 108x_2x_3x_7^2x_8^2x_9^2 + 324x_3x_4^2x_6x_7x_8x_9^2 \\
&+ 72x_3x_5x_6^3x_9^3 - 216x_3x_5x_6^2x_8x_9^3 + 216x_3x_5x_6x_8^2x_9^3 \\
&- 72x_3x_5x_8^3x_9^3 - 216x_4x_6x_7^2x_8^2x_9^2 - 216x_3x_4x_6^2x_9^3
\end{aligned}
$$

$$
\begin{aligned}
&+432x_3x_4x_6x_8x_9^3-216x_3x_4x_8^2x_9^3+108x_5x_6^2x_7x_8x_9^2\\
&-144x_5x_6x_7x_8^2x_9^2+36x_5x_7x_8^3x_9^2+432x_6^2x_7x_8x_9^3\\
&-720x_6x_7x_8^2x_9^3+288x_7x_8^3x_9^3+108x_2x_3x_7x_8x_9^2\\
&+108x_3x_4^2x_5x_6x_9-108x_3x_4^2x_6x_9^2+216x_4x_6x_7x_8x_9^2\\
&+36x_3x_4^3x_9-36x_5x_6^2x_9^2+72x_5x_6x_8x_9^2-36x_5x_8^2x_9^2\\
&-72x_6^2x_9^3+144x_6x_8x_9^3-72x_8^2x_9^3\\
&-36x_2x_3x_9^2-72x_4x_6x_9^2.
\end{aligned}
$$

$$\{x_1,\{x_5,x_9\}\}+\{x_9,\{x_1,x_5\}\}+\{x_5,\{x_9,x_1\}\}=0.$$