Duality in Finite Element Exterior Calculus

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Finite element exterior calculus

Triangulate the domain into simplices. On a simplex $T$, we have spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^\perp\Lambda^k(T)$ of $k$-forms on $T$ with polynomial coefficients of degree at most $r$.

Special cases

► scalar fields
  ► Lagrange
  ► Discontinuous Galerkin

► vector fields
  ► Brezzi–Douglas–Marini elements
  ► Raviart–Thomas elements
  ► Nédélec elements

Example

In three dimensions, $\mathcal{P}_r\Lambda^1(T)$ and $\mathcal{P}_r^\perp\Lambda^1(T)$ are Nédélec $H(\text{curl})$ elements of the 2nd and 1st kinds, respectively.

See (Arnold, Falk, Winther, 2006).
Duality: a motivating example

Let \( \Omega \) be an 3-dimensional domain. Given \( \alpha \in \Lambda^1(\Omega) \) and \( \beta \in \Lambda^2(\Omega) \), we can compute

\[
\int_\Omega \alpha \wedge \beta.
\]

Integration is a perfect pairing \( \Lambda^1(\Omega) \times \Lambda^2(\Omega) \rightarrow \mathbb{R} \).

- For any nonzero \( \alpha \in \Lambda^1(\Omega) \), there exists a \( \beta \in \Lambda^2(\Omega) \) such that \( \int_\Omega \alpha \wedge \beta > 0 \), and vice versa.

In this setting, given \( \alpha \), it is easy to construct such a dual \( \beta \). If \( \alpha = \alpha_x \, dx + \alpha_y \, dy + \alpha_z \, dz \), then we can set

\[
\beta = \alpha_x \, dy \wedge dz + \alpha_y \, dz \wedge dx + \alpha_z \, dx \wedge dy = *\alpha.
\]

- \( \int_\Omega \alpha \wedge \beta = \int_\Omega (\alpha_x^2 + \alpha_y^2 + \alpha_z^2) \, d\text{vol} > 0. \)
- \( \beta \) only depends on \( \alpha \) pointwise.
Duality in finite element exterior calculus

Let $T$ be a simplex. Given $\alpha \in \Lambda^k(T)$ and $\beta \in \Lambda^{n-k}(T)$, we consider the pairing

$$(\alpha, \beta) \mapsto \int_T \alpha \wedge \beta.$$ 

Arnold, Falk, and Winther show that integration is a perfect pairing in the two settings

$$\mathcal{P}_r^\perp \Lambda^k(T) \times \mathcal{P}_{r+k} \Lambda^{n-k}(T) \rightarrow \mathbb{R},$$

$$\mathcal{P}_r \Lambda^k(T) \times \mathcal{P}_{r+k+1}^\perp \Lambda^{n-k}(T) \rightarrow \mathbb{R}.$$ 

- $\mathcal{P}$ denotes forms with vanishing tangential trace on $\partial T$.

Problem

Given $\alpha \in \mathcal{P}_r \Lambda^k(T)$, find a dual $\beta \in \mathcal{P}_{r+k+1}^\perp \Lambda^{n-k}(T)$ such that

- $\int_T \alpha \wedge \beta > 0$, and
- $\beta$ only depends on $\alpha$ pointwise.
The simplex

To illustrate, focus on dim $T = 2$. The standard simplex $T$ sits inside the first orthant $O$ as those points that satisfy $x + y + z = 1$.

Key ideas

- Identify $\mathcal{P}_r \Lambda^k(T)$ and $\mathcal{P}_r^{-} \Lambda^k(T)$ with spaces $\mathcal{P}_r \Lambda^k(O)$ and $\mathcal{P}_r^{-} \Lambda^k(O)$ of differential forms on $O$.
- Exploit a natural duality relationship between the $\mathcal{P}$ and $\mathcal{P}^-$ spaces.
**Vertical and horizontal antisymmetric tensors**

Let $E$ be a vector space, let $H \subset E$ be a hyperplane, and let $X$ be a vector not in the hyperplane. To illustrate, focus on $\text{dim } E = 3$.

\[
\begin{array}{c}
\text{X} \\
\text{H}
\end{array}
\]

- Choose a basis for $E^* = \langle e^1, e^2, e^3 \rangle$ so that $e^3(Y) = 0$ for all $Y \in H$ and $e^1(X) = e^2(X) = 0$.
- This splitting of $E^*$ extends to a splitting of $\Lambda^\bullet E^*$ into vertical and horizontal subspaces $(\Lambda^\bullet E^*)^\perp$ and $(\Lambda^\bullet E^*)^\top$.

<table>
<thead>
<tr>
<th>$\Lambda^k E^*$</th>
<th>vertical</th>
<th>horizontal</th>
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<tbody>
<tr>
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<tr>
<td>$\Lambda^2 E^*$</td>
<td>$\langle e^1 \wedge e^3, e^2 \wedge e^3 \rangle$</td>
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Note that

\[
\Lambda^k H^* \cong (\Lambda^{k+1} E^*)^\perp, \quad \Lambda^k H^* \cong (\Lambda^k E^*)^\top.
\]
Vertical and horizontal differential forms

Let \( x = (x, y, z) \in T \). Apply the above discussion \( E = \mathbb{R}^3 = T_xO \), \( H = T_xT \), \( e^3 = dx + dy + dz \), and \( X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \).

Definition

Let \( P_r \Lambda^k(O) \) denote those \((k + 1)\)-forms on \( O \) that

- are \textbf{vertical} at every point \( x \in T \), and
- whose coefficients are homogeneous polynomials of degree \( r \).

Let \( P_r^{-} \Lambda^k(O) \) denote those \( k \)-forms on \( O \) that

- are \textbf{horizontal} at every point \( x \in T \), and
- whose coefficients are homogeneous polynomials of degree \( r \).

Theorem

\[ P_r \Lambda^k(T) \cong P_r \Lambda^k(O), \quad P_r^{-} \Lambda^k(T) \cong P_r^{-} \Lambda^k(O) \]
Duality

Problem (reframed)

Given \( \alpha \in P_r \Lambda^k(O) \), find a dual \( \beta \in \hat{P}_{r+k+1} \Lambda^{n-k}(O) \) such that

\[ \int_T \alpha \wedge \beta > 0, \text{ and} \]

\( \beta \) only depends on \( \alpha \) pointwise.

Theorem

We explicitly construct such a map \( P_r \Lambda^k(O) \rightarrow \hat{P}_{r+k+1} \Lambda^{n-k}(O) \).

Example

\[ \begin{align*}
\quad & \quad \text{Let dim } T = 2, \text{ and let } \alpha \in P_r \Lambda^1(O), \text{ a vertical 2-form on } O. \\
\quad & \quad \text{Write } \alpha = \alpha_x \, dy \wedge dz + \alpha_y \, dz \wedge dx + \alpha_z \, dx \wedge dy. \\
\quad & \quad \text{Set } \beta = \alpha_x yz \, dx + \alpha_y zx \, dy + \alpha_z xy \, dz. \\
\quad & \quad \text{Then } \beta \text{ is horizontal, has vanishing tangential trace on the boundary, and has coefficients of degree } r + 2. \\
\quad & \quad \alpha \wedge \beta = (\alpha_x^2 yz + \alpha_y^2 zx + \alpha_z^2 xy) \, dvol, \text{ a positive multiple of } dvol \text{ on the interior.}
\end{align*} \]
Thank you
Vertical and horizontal antisymmetric tensors

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Characterizations of $\alpha$ being vertical.

- $\alpha \wedge e^3 = 0$.
- $\alpha$ is of the form $\gamma \wedge e^3$ for some $\gamma$.
- The restriction of $\alpha$ to $H$ is zero.

Characterizations of $\beta$ being horizontal.

- $i_X \beta = 0$.
- $\beta = i_X \gamma$ for some $\gamma$.
- $\beta$ is orthogonal to all vertical tensors.