Yang-Mills Replacement

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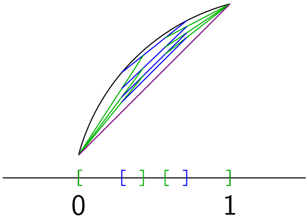
14 November, 2015

Schwarz Alternating Method

Example

Let $f\colon [0,1] \to \mathbb{R}$. We want to make f harmonic while fixing its boundary values.



- By locally replacing f with a harmonic function, we get a global harmonic function in the limit.

- Colding and Minicozzi (2008) locally replace maps $u\colon \Sigma^2 \to M$ with harmonic maps, with bounds.
- I showed that one can similarly locally replace connections on 4-manifolds with Yang-Mills connections, with bounds.

Applications

- Colding and Minicozzi used harmonic replacement to prove finite extinction time of Ricci flow on homotopy 3-spheres.
  - They construct a sweep-out of the 3-sphere by immersed 2-spheres and “tighten” each 2-sphere using harmonic replacement.
- Yang-Mills replacement could relate the topology of the moduli space of anti-self-dual Yang-Mills connections to the topology of all connections modulo gauge.
  - Taubes, *Stable Topology* (1989).
  - Donaldson invariants.
  - Perform Yang-Mills replacement on connections in a compact family representing a homotopy or homology class.
- Yang-Mills replacement has parallels with Yang-Mills gradient flow.
  - Ability to choose balls gives more control.

Harmonic Maps and Yang-Mills Connections

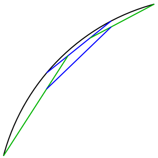
Harmonic maps	Yang-Mills connections
$u\colon \Sigma \to M \subseteq \mathbb{R}^N$	Connection A on a principal G-bundle $P \to X$
u is an $\mathbb{R}^N$-valued 0-form on Σ.	Locally, $A = d + a$, a is a $\mathfrak{g}$-valued 1-form on X.
du is an $\mathbb{R}^N$-valued 1-form.	$F_A = da + \frac{1}{2}[a \wedge a]$ is a $\mathfrak{g}$-valued 2-form.
Energy $= \frac{1}{2}\int_\Sigma \lvert du \rvert^2$	Energy $= \frac{1}{2}\int_X \lvert F_A \rvert^2$
Invariant under conformal change of metric if $\dim \Sigma = 2$	Invariant under conformal change of metric if $\dim X = 4$
$(\Delta u)^\top = (d^* du)^\top = 0.$	$d_A^* F_A = 0.$

The Dirichlet Problem

- To locally replace a connection with a Yang-Mills connection, we must solve the Dirichlet problem.
- On B^4, for "small" boundary data A_∂ on ∂B^4, we must solve:

$$\begin{aligned} d_A^* F_A = 0 \quad & \text{on } B^4 \\ i^* A = A_\partial \quad & \text{on } \partial B^4 \end{aligned}$$

- Solved by Marini (1992) for smooth boundary values.



- Our boundary values are $L^2_{1/2}(\partial B^4)$, and solutions are $L^2_1(B^4)$.

Local Yang-Mills Replacement

Theorem (YBK)

- *For any* $L^2_1(B^4)$ *low-energy connection* A*, there exists a low-energy* $L^2_1(B^4)$ *Yang-Mills connection* B*, unique up to gauge, such that* $i^*A = i^*B$*.*
- *There are* $\tilde{A}$ *and* $\tilde{B}$*, gauge equivalent by an* $L^2_2(B^4)$ *gauge transformation to* A *and* B*, respectively, such that*

$$\left\|\tilde{A} - \tilde{B}\right\|^2_{L^2_1(B^4)} \leq C\left(\|F_A\|^2_{L^2(B^4)} - \|F_B\|^2_{L^2(B^4)}\right).$$

- *The linear interpolation between* A *and* B *has monotone decreasing energy.*
  - *Equality if and only if* A *is already Yang-Mills.*

Linearization

To solve the Dirichlet problem, we want to invert the map $A \mapsto (d_A^* F_A, i^* A)$ near the trivial connection using the inverse function theorem.

Harmonic maps	Yang-Mills connections
$u \colon B^2 \to M \subseteq \mathbb{R}^N$	Connection $A = d + a$ on a principal G-bundle $P \to B^4$
u is an $\mathbb{R}^N$-valued 0-form.	a is a $\mathfrak{g}$-valued 1-form.
$(d^* du)^\top$	$d_A^* F_A$
Linearize near $u =$ constant.	Linearize near $a = 0$.
$d^* d\phi = \Delta\phi$	$d^* d\alpha \neq \Delta\alpha = d^* d\alpha + dd^*\alpha$
$\phi \mapsto (d^* d\phi, i^*\phi)$ is invertible.	$\alpha \mapsto (d^* d\alpha, i^*\alpha)$ is not invertible.
	Solution: Can get $d^*\alpha = 0$ by choosing a good gauge.

Gauge Fixing

Gauge transformations, that is, automorphisms of $P \to B^4$, act on connections on B^4.

- Energy, and hence the Yang-Mills equations, are invariant under gauge transformations.

Theorem (Dirichlet Uhlenbeck gauge fixing, YBK)

Any low-energy $L^2_1(B^4)$ connection A is gauge equivalent to an $L^2_1(B^4)$ connection $\tilde{A} = d + \tilde{a}$ such that:

- *$\tilde{A}$ is in Dirichlet Coulomb gauge, that is,*
  - *$d^*\tilde{a} = 0$ on B^4, and*
  - *$d^*_{\partial B^4} i^*\tilde{a} = 0$ on ∂B^4.*
- $\|\tilde{a}\|_{L^2_1(B^4)} \leq C \|F_A\|_{L^2(B^4)}$.

The boundary condition $d^*_{\partial B^4} i^*\tilde{a} = 0$ is preserved under gauge transformations satisfying Dirichlet boundary conditions.

Solving the Dirichlet Problem

We want to invert the map $A \mapsto (d_A^* F_A, i^* A)$ near the trivial connection, where A is an $L_1^2(B^4)$ connection.

- The linearization is $\alpha \mapsto (d^* d\alpha, i^*\alpha)$, which is not invertible.
- Gauge fixing lets us assume $d^*\alpha = 0$.
  - The linearization is now equal to $\alpha \mapsto (\Delta\alpha, i^*\alpha)$.
- $\alpha \mapsto (\Delta\alpha, i^*\alpha)$ is still not invertible on 1-forms.
  - Dirichlet boundary conditions for the Hodge Laplacian require specifying $i^*\alpha$ and $i^* d^*\alpha$.
  - $\alpha \mapsto (\Delta\alpha, i^*\alpha, i^* d^*\alpha)$ is invertible, but only for $\alpha \in L_2^2(B^4)$.
- Restricting to $\ker d^*$ gives an isomorphism

  $$(d^* d, i^*) \colon L_2^2(B^4) \cap \ker d^* \to L^2(B^4) \cap \mathrm{range}(d^*) \times L_{3/2}^2(\partial B^4).$$

- But in the regularity we want,

  $$(d^* d, i^*) \colon L_1^2(B^4) \cap \ker d^* \to L_{-1}^2(B^4) \cap \mathrm{range}(d^*) \times L_{1/2}^2(\partial B^4)$$

  is not injective.
- Solution: Use a target space slightly larger than $L_{-1}^2(B^4)$.

Solving the Dirichlet Problem

Choosing the Target Banach Space

We want to invert the map $A \mapsto (d_A^* F_A, i^* A)$ near the trivial connection, where A is an $L_1^2(B^4)$ connection.

- The linearization is $\alpha \mapsto (d^* d\alpha, i^* \alpha)$.

Definition

$$\begin{array}{ccc} & & \text{Dual} \\ L_1^2(B^4)^0 = \{\alpha \in L_1^2(B^4) \mid \alpha|_{\partial B^4} = 0\} & & L_{-1}^2(B^4) \\ \big\downarrow & & \big\uparrow \\ L_1^2(B^4)^{\mathrm{rel}} = \{\alpha \in L_1^2(B^4) \mid i^* \alpha = 0\} & & L_{-1}^2(B^4)^{\mathrm{rel}} \end{array}$$

- $d^* d$ is bounded as an operator $d^* d \colon L_1^2(B^4) \to L_{-1}^2(B^4)$.
- $d^* d$ is still bounded as $d^* d \colon L_1^2(B^4) \to L_{-1}^2(B^4)^{\mathrm{rel}}$.
- $\alpha \mapsto (d^* d\alpha, i^* \alpha)$ is invertible as an operator

$$(d^* d, i^*) \colon L_1^2(B^4) \cap \ker d^* \to L_{-1}^2(B^4)^{\mathrm{rel}} \cap \mathrm{range}(d^*) \times L_{1/2}^2(\partial B^4).$$

Solving the Dirichlet Problem

Projecting to range(d^*)

We want to invert the map $A \mapsto (d_A^* F_A, i^* A)$ near the trivial connection, where A is an $L_1^2(B^4)$ connection.

- The linearization $\alpha \mapsto (d^* d\alpha, i^*\alpha)$ is invertible as an operator

$$(d^* d, i^*) \colon L_1^2(B^4) \cap \ker d^* \to L_{-1}^2(B^4)^{\mathrm{rel}} \cap \operatorname{range}(d^*) \times L_{1/2}^2(\partial B^4).$$

- Problem: $d_A^* F_A$ does not lie in range(d^*) in general.
- Solution: Project to range(d^*).
  - Let π_{d^*} be the $L^2(B^4)$-projection to range(d^*).
  - π_{d^*} extends to a bounded operator $L_{-1}^2(B^4)^{\mathrm{rel}} \to L_{-1}^2(B^4)^{\mathrm{rel}}$.
  - The linearization of $A \mapsto (\pi_{d^*} d_A^* F_A, i^* A)$ at the trivial connection is $(\pi_{d^*} d^* d\alpha, i^*\alpha) = (d^* d\alpha, i^*\alpha)$.
- Given A_∂ small in the $L_{1/2}^2(\partial B^4)$ norm, we can solve

$$\begin{aligned} \pi_{d^*} d_A^* F_A &= 0 \quad \text{on } B^4 \\ i^* A &= A_\partial \quad \text{on } \partial B^4 \end{aligned}$$

- We also have $d^* a = 0$ and that a is small in $L_1^2(B^4)$.

Solving the Dirichlet Problem

Concluding that the connection minimizes energy

- We have found a $B = d + b$ such that $\pi_{d^*} d_B^* F_B = 0$ and b is small in $L_1^2(B^4)$.
- We want to conclude that $d_B^* F_B = 0$.
- In higher regularity $b \in L_2^2(B^4)$, given $\pi_{d^*} d_B^* F_B = 0$, we can prove an inequality of the form

$$\|d_B^* F_B\|_{L^2(B^4)} \leq C \|b\|_{L^4(B^4)} \|d_B^* F_B\|_{L^2(B^4)} .$$

  - Conclude that $d_B^* F_B = 0$ as long as $\|b\|_{L^4(B^4)}$ is small.
- This argument fails at $b \in L_1^2(B^4)$ regularity.
- Instead, we directly show that B locally minimizes energy and is thus Yang-Mills, using the inequality

$$\|A - B\|_{L_1^2(B^4)}^2 \leq C \left(\|F_A\|_{L^2(B^4)}^2 - \|F_B\|_{L^2(B^4)}^2 \right) .$$

  - The inequality holds even if B only satisfies $\pi_{d^*} d_B^* F_B = 0$, along with assumptions of small energy, matching on the boundary, and Dirichlet Coulomb gauge.

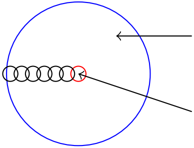
Towards Global Yang-Mills Replacement

We want to repeat Yang-Mills replacement on balls covering the manifold X to obtain a global Yang-Mills connection in the limit.

Bubbling

- Yang-Mills replacement requires small energy on each ball.
- We can guarantee this initially by choosing small enough balls.
- Yang-Mills replacement on one ball might concentrate energy in another ball.

Replacement could move energy inward.



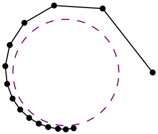
- Potential solution: Moving energy costs energy.

$$\left\| \tilde{A} - \tilde{B} \right\|^2_{L^2_1(B^4)} \leq C \left(\|F_A\|^2_{L^2(B^4)} - \|F_B\|^2_{L^2(B^4)} \right).$$

Towards Global Yang-Mills Replacement

We want to repeat Yang-Mills replacement on balls covering the manifold X to obtain a global Yang-Mills connection in the limit.

Limit cycles in the space of connections



- Differences must go to zero by

$$\left\| \tilde{A} - \tilde{B} \right\|_{L_1^2(B^4)}^2 \leq C \left(\|F_A\|_{L^2(B^4)}^2 - \|F_B\|_{L^2(B^4)}^2 \right).$$

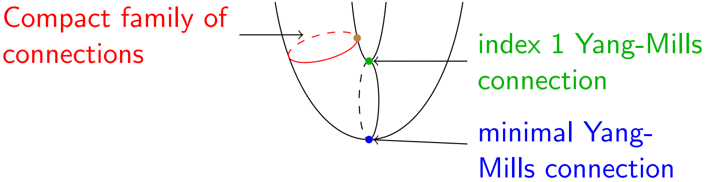
  - Not strong enough to guarantee convergence.
- Can still use weak subsequence convergence.
  - The limiting global Yang-Mills connection will not depend continuously on the initial connection.
- Łojasiewicz inequality.

Towards Global Yang-Mills Replacement

We want to repeat Yang-Mills replacement on balls covering the manifold X to obtain a global Yang-Mills connection in the limit.

- Given a compact family of connections, we can choose the sequence of balls uniformly for the entire family.
- Ideally, the limiting Yang-Mills connection will depend continuously on the initial connection.

Yang-Mills connections with positive Morse index



- Global Yang-Mills replacement cannot be continuous in the initial data.
  - Might be continuous if the initial data is below all non-minimal critical points.

Thank You

Acknowledgments

- Tom Mrowka
- National Science Foundation
- Department of Defense, NDSEG

Selected References

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Towards Global Yang-Mills Replacement

Bonus Slide

We want to repeat Yang-Mills replacement on balls covering the manifold X to obtain a global Yang-Mills connection in the limit.

Discontinuous normal components

- Only the tangential components of the replacement match the original connection on ∂B^4.
- The normal derivative of the normal component of the new connection is not $L^2(X)$ across ∂B^4.
- After local Yang-Mills replacement, the global connection is no longer $L^2_1(X)$.
- Solution: With a different choice of gauge on a slightly larger ball, the connection becomes $L^2_1(X)$.