

Topology I Final Exam

December 21, 2016

Name: _____

- There are ten questions, each worth ten points, so you should pace yourself at around 10–12 minutes per question, since they vary in difficulty and you'll want to check your work.
- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
 - Make a note on the printed page that your work continues on the back of the previous page.
 - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. (a) (4 points) Define when a topological space X is compact.

(b) (6 points) Prove that a closed subset of a compact space is compact.

2. (a) (2 points) Let $f: X \rightarrow Y$ be a function between topological spaces X and Y . Define continuity. The function f is continuous if ...

(b) (2 points) State the extreme value theorem for a map $f: X \rightarrow \mathbb{R}$.

(c) (6 points) Prove the extreme value theorem.

Hints: The first part of the proof uses an earlier result about general maps $f: X \rightarrow Y$. You should state this result, but you can use it without proof. The second part of the proof uses a fact about subsets A of \mathbb{R} . You should state and prove this fact. Hint: Consider $\{(-\infty, a) \mid a \in A\}$.

3. (a) (2 points) Define when a topological space X is connected.
- (b) (4 points) Let $f: X \rightarrow Y$ be a surjective continuous map between topological spaces X and Y . Show that if X is connected, then Y is also connected.
- (c) (2 points) What part of your proof fails if f is not surjective?
- (d) (2 points) Provide a counterexample. That is, construct an example of two topological spaces X and Y and a continuous function $f: X \rightarrow Y$, where X is connected but Y is not. Make sure to show that, in your example, f is continuous, X is connected, and Y is not connected.

4. (a) (2 points) Define when a topological space X is Hausdorff.

(b) (6 points) Prove that a compact subset of a Hausdorff space is closed.

(c) (2 points) Provide a counterexample when the space X is not Hausdorff. That is, give an example of a topological space X and a subset A . Show that X is not Hausdorff, that A is compact, but that A is not closed.

5. (a) (2 points) Let X be a set. Define a basis for a topology on X . That is, what are the axioms for a collection \mathcal{B} to be a basis?
- (b) (2 points) Define the topology on X generated by the basis \mathcal{B} .
- (c) (2 points) Let X be a set. Define a metric on X .
- (d) (4 points) Let d be a metric on the set X . Define the metric topology on X induced by d using a basis. Prove that this basis satisfies the basis axioms.

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6. (a) (2 points) Let X and Y be topological spaces. Define the product topology on $X \times Y$ using a basis. Check that the basis satisfies the basis axioms.
- (b) (2 points) Let X be a topological space. Show that a subset A of X is open if and only if for every $a \in A$, there exists an open set U such that $a \in U \subseteq A$.
- (c) (6 points) Let X be a topological space, and let $\Delta = \{(x, y) \in X \times X \mid x = y\}$ be the diagonal in $X \times X$. Show that X is Hausdorff if and only if Δ is closed in $X \times X$.

7. Let X and Y be topological spaces, let $\pi_X: X \times Y \rightarrow X$ and $\pi_Y: X \times Y \rightarrow Y$ be the projections, and let W be a subset of $X \times Y$. Consider the following two statements:

- If W is open in $X \times Y$, then $\pi_X(W)$ and $\pi_Y(W)$ are open in X and Y , respectively.
- If $\pi_X(W)$ and $\pi_Y(W)$ are open in X and Y , respectively, then W is open in $X \times Y$.

(a) (4 points) Prove the statement that is true.

(b) (2 points) For the statement that is false, provide a counterexample for the case of $X = Y = \mathbb{R}$. That is, construct a subset W of $\mathbb{R} \times \mathbb{R}$, and show that it contradicts the statement that you have determined to be false.

(c) (4 points) Apart from being about product spaces, this question is unrelated to the above. Let $f: Z \rightarrow X$ and $g: Z \rightarrow Y$ be continuous functions. Define $h: Z \rightarrow X \times Y$ by $h(z) = (f(z), g(z))$. Show that h is continuous with respect to the product topology on $X \times Y$.

8. (a) (2 points) Let X be a topological space. Let A be a subset of X . Define the closure \overline{A} of A .
- (b) (2 points) Show that $x \in \overline{A}$ if and only if any open neighborhood U of x intersects A nontrivially.
- (c) (2 points) Define when a topological space X is first-countable.
- (d) (4 points) Let A be a subset of a first-countable topological space X . Show that $x \in \overline{A}$ if and only if x is the limit of a sequence (a_n) , where $a_n \in A$ for all n .

9. This problem is about the uniform metric on $[0, 1]^{\mathbb{N}}$. Technically, the study guide talked about $\mathbb{R}^{\mathbb{N}}$. If you'd like, you can do these problems for $\mathbb{R}^{\mathbb{N}}$ instead, but it will be more work.
- (a) (4 points) Define the uniform metric on $[0, 1]^{\mathbb{N}}$. You do not need to prove that it is a metric.
- (b) (6 points) Let $[0, 1]^{\infty}$ be the set of all sequences in $[0, 1]^{\mathbb{N}}$ that are eventually zero. That is, $\mathbf{x} = (x_i) \in [0, 1]^{\infty}$ if all but finitely many of the x_i are zero. Compute, with proof, the closure of $[0, 1]^{\infty}$ in $[0, 1]^{\mathbb{N}}$ with respect to the uniform topology.

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10. (a) (2 points) Define when a topological space X is locally compact.
- (b) (2 points) Let $Y = X \cup \{\infty\}$ be the one-point compactification of X . Define the topology on Y in terms of the topology on X .
- (c) (3 points) In class, we saw that the one-point compactification of \mathbb{R} was the circle. Let X be the half-open interval $[0, 1)$. Check that it is locally compact Hausdorff, and determine its one-point compactification. (You may use without proof the fact that closed and bounded subsets of \mathbb{R} are compact.)
- (d) (3 points) Let X be the closed interval $[0, 1]$. Check that it is locally compact Hausdorff, and determine its one-point compactification.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	