TOPOLOGY HOMEWORK 1, DUE SEPTEMBER 7

Instructions

(1) Write down the names of the people you worked with.
(2) Write down any resources you used other than ones that most of your classmates would be familiar with, such as the textbook or Wikipedia.
(3) Write down the number of hours it took you to complete this assignment.
(4) Write your name, Math 4171, and the homework number.
(5) Hand in your homework in class.

Problems

(1) Read the syllabus. Check that you don't have any conflicts with the in-class midterm dates on the course calendar on the course webpage. If you have any conflicts, please send me an email, and we'll try to schedule something that works for everyone. If you have no conflicts, write down the dates of the two midterms, and write that you don't have conflicts with them.
(2) Do Exercise 13.1 on page 83 of Munkres.
(3) Prove that if a topological space X is Hausdorff, then a sequence x_i in X converges to at most one point in X. Note: You can find the relevant definitions on page 98 of Munkres, and they're the same as the ones I gave in class. You can also find this claim and its proof on page 99, but, in order to get the practice you need in order to do the other problems on this problem set, I'd strongly recommend not looking at it and writing your own proof.
(4) In the next few problems, we'll look at some weird topologies in order to get a feel for the definitions. Most topologies that we see in real life are not that weird, so weird topologies tend to only be very helpful from a pedagogical standpoint. Having said that, in the mid-20th century, a non-Hausdorff topology revolutionized algebra, geometry, *and* number theory, so you never know.
   (a) In class, we described a topology by using open sets, and then we defined closed sets to be their complements. We could just as easily describe a topology by specifying its closed sets, and then letting the open sets be their complements.
   Let X be a set, and consider the topology where a subset C of X is closed if either $C = X$ or C is finite. This topology is called the *cofinite topology* on X. Note: the empty set is finite.
   Look at Example 3 on page 77, and note that we've described using closed sets the same topology that Munkres describes using open sets. Using open sets, Munkres shows that the cofinite topology satisfies the axioms of a topology. In this problem, you will prove that the cofinite topology satisfies the axioms of a topology using closed sets. Begin by reading and understanding Theorem 17.1, its proof, the paragraph immediately following the proof, and why the axioms for closed sets in Theorem 17.1 are equivalent to the axioms for open sets on page 76. (It's about half a page. You don't need to write anything down to demonstrate that you did the reading. I trust you.)
   Show that the cofinite topology, as described above, satisfies the three axioms in Theorem 17.1.

(b) Show that, with the cofinite topology, X is Hausdorff if and only if it is finite.

(5) (a) Let $X = \mathbb{R}$. We will consider a nonstandard collection of open sets on $\mathbb{R}$ with fewer open sets than usual. Let U be open if it is either empty, all of $\mathbb{R}$, or the ray (a, ∞) for some $a \in \mathbb{R}$. This topology is called the *right order topology* on $\mathbb{R}$. Show that this decision of which sets to call open satisfies the axioms of a topology discussed in class and on page 76 of Munkres.

Note: You'll need the concept of *infimum*, which we haven't yet covered in class. See page 27 or me in office hours if you haven't seen it in a previous course.

(b) Using the definition of Hausdorff, show that X is not Hausdorff with this topology.

(c) Show that x is a limit of the constant sequence $(y, y, y, \dots)$ if and only if $x \leq y$. It is an understatement to say that this sequence has more than one limit.

(6) Do exercise 13.4(a) on page 83 of Munkres.

Hint: This problem is easy but quite abstract. In other words, it's easy only in retrospect. Note that in the problem set up, you're not taking an intersection of the open sets themselves. You're taking an intersection of the collections of open sets. To get a feel for the problem, imagine that $X = \mathbb{R}$, and think about the standard topology on $\mathbb{R}$, the cofinite topology on $\mathbb{R}$, the right order topology on $\mathbb{R}$, and imagine that they've also brought their oddball topology friends. You pick a couple of them, and then take only the open sets that they all have in common. Is that a valid topology?

Hint: For the second part, as you might guess from the phrasing, the answer is no. However, you must demonstrate a counterexample. Consider the right order topology, along with the *left order topology*, which is defined the same way but with $(-\infty, a)$ in place of (a, ∞).

Remark: You do not need to write down a proof that the left order topology satisfies the axioms of a topology. If you had to, you could write, "The proof is analogous to the proof for the right order topology," and leave it at that. However, it may be useful for the future to think about the map $\mathbb{R} \to \mathbb{R}$ that sends x to $-x$. Notice that this map sends open sets in the right order topology to open sets in the left order topology, and vice versa. Note also that this map preserves unions and intersections. Thus, using this map, you can see that the left order topology satisfies the axioms of a topology as a *consequence* of the fact that the right order topology satisfies these axioms.