TOPOLOGY HOMEWORK 11, DUE NOVEMBER 16

(1) Consider $\mathbb{R}^{\mathbb{N}}$ with the *box* topology.

(a) Let $\mathbf{x}_n$ be a sequence of points in $\mathbb{R}^{\mathbb{N}}$ such that there are constants N and I such that if $i > I$ and $n > N$, then $(\mathbf{x}_n)_i = 0$, and such that the coordinate sequences converge to zero for all i, that is, $\lim_{n\to\infty}(\mathbf{x}_n)_i = 0$. In other words, $\mathbf{x}_n$ looks something like this:

$$\begin{array}{rccccccc} \mathbf{x}_1 = & (x_{11}, & x_{12}, & \dots, & x_{1I}, & x_{1,I+1}, & x_{1,I+2}, & \dots), \\ \mathbf{x}_2 = & (x_{21}, & x_{22}, & \dots, & x_{2I}, & x_{2,I+1}, & x_{2,I+2}, & \dots), \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{x}_N = & (x_{N1}, & x_{N2}, & \dots, & x_{NI}, & x_{N,I+1}, & x_{N,I+2}, & \dots), \\ \mathbf{x}_{N+1} = & (x_{N+1,1}, & x_{N+1,2}, & \dots, & x_{N+1,I}, & 0, & 0, & \dots), \\ \mathbf{x}_{N+2} = & (x_{N+2,1}, & x_{N+2,2}, & \dots, & x_{N+2,I}, & 0, & 0, & \dots), \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Show that $\mathbf{x}_n$ converges to zero in $\mathbb{R}^{\mathbb{N}}$ with the box topology.

(b) Consider now the sequence

$$\begin{array}{rccccccc} \mathbf{x}_1 = & (a_1, & 0, & 0, & 0, & 0, & 0, & \dots), \\ \mathbf{x}_2 = & (0, & a_2, & 0, & 0, & 0, & 0, & \dots), \\ \mathbf{x}_3 = & (0, & 0, & a_3, & 0, & 0, & 0, & \dots), \\ \mathbf{x}_4 = & (0, & 0, & 0, & a_4, & 0, & 0, & \dots), \\ \mathbf{x}_5 = & (0, & 0, & 0, & 0, & a_5, & 0, & \dots), \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{array}$$

Assume that all of the a_n are nonzero. Show that the sequence $\mathbf{x}_n$ does not converge to zero, no matter what the a_n are (even if they converge to zero, for example).

Note: The upshot of this problem is that it shows that the sequences of the type in part 1a are the only sequences that converge to zero in the box topology. If we have a sequence and replace terms with zeroes, it only makes it more likely to converge to zero. If we remove "rows" and "columns" from the sequence, it also only makes it more likely to converge. It is not hard to show that if you start with a sequence that is *not* of the type in part 1a, then via these operations you can convert to a sequence of the type in this part 1b, which does not converge, so the original sequence doesn't either. There's also nothing special about zero, so we can think of convergence in the box topology as requiring that the coordinate sequences eventually be eventually constant.

(c) Under what conditions does the sequence in 1b converge in the uniform topology on $\mathbb{R}^{\mathbb{N}}$?

(2) Do problem 20.6 on page 127.

(3) Do problem 20.7 on page 127. Theorem 21.1, the ε–δ definition of continuity, may be helpful. We proved a while ago that on $\mathbb{R}^n$, the ε–δ definition of continuity is equivalent to the topological definition, but the only fact we actually used was that $\mathbb{R}^n$ is a metric space.

(4) Do problem 20.8 on pages 127–128. For (b), I recommend finding sequences in $\mathbb{R}^\infty$ that converge in one topology but not the other. For part (c), the same idea can be used to show topologies are different. To show topologies are the same, Lemma 16.1 gives you bases on H that you can use.

(5) Do problem 20.9 on page 128. Turn this one in with homework 12.
(6) Do problem 20.10 on page 128. Turn this one in in with homework 12.