TOPOLOGY HOMEWORK 14, DUE DECEMBER 7

(1) Do problem 26.1 on page 170. You can do this problem directly or use the fact that the continuous image of a compact set is compact.

(2) Do problem 26.2 on page 171. This problem is easier to do with the characterization of compactness in terms of closed sets that we will cover on Friday.

(3) Do problem 26.3 on page 171.

(4) Do problem 26.4 on page 171. In class, we showed that for subsets of $\mathbb{R}^n$, being compact is equivalent to being closed and bounded. One direction easily generalizes to general metric spaces. The other direction is false for general metric spaces.

A counterexample that's relevant in mathematics is the closed ball of unit radius in an infinite-dimensional space like $\mathbb{R}^{\mathbb{N}}$ with the uniform metric or ℓ^2 with the ℓ^2 metric. That is, the closed ball is the set $\{x \mid d(\mathbf{0}, x) \leq 1\}$. This set is obviously bounded, and it is the preimage under the continuous map $x \mapsto d(\mathbf{0}, x)$ of the closed set $[0, 1]$, hence closed. The fact that the closed unit ball is not compact causes issues. In many cases in analysis, these issues can be resolved by defining a coarser topology called the *weak topology*. The weak topology has the downside of not being metrizable, but the upside that closed unit ball (defined using the original metric) is compact. Consequently, in many problems, we need to be thinking about both the metric topology and the weak topology.

However, for this problem, there is a simpler counterexample. It turns out that, moreover, there is an isometric embedding of this counterexample onto a closed subset of the unit ball above, which proves that the unit ball is not compact by the contrapositive of Theorem 26.2. (Constructing this embedding is not hard, but is not part of the homework.)

(5) Do problem 26.5 on page 171. We talked about the outline of this proof in class, but you should fill in the details. Make sure to save yourself some work by using Lemma 26.4.

(6) Do problem 26.6 on page 171. You'll need a fact from next lecture (Friday) for this one.

(7) Do problem 26.7 on page 171. Warning: Don't assume that open sets or closed sets are rectangles. Small hint: Tube lemma. A picture might help you figure out how to approach the problem.

(8) Do problem 26.8 on page 171. I believe the hint is unnecessarily complicated. Show f is continuous by showing that the preimage of a closed set C is closed, using the fact that the intersection of G_f with $X \times C$ is closed and applying the previous exercise. A picture may help you get a feel for what's going on.

(9) (a) Construct a counterexample to problem 26.8 when Y is not compact by finding a discontinuous function $f \colon \mathbb{R} \to \mathbb{R}$ whose graph is closed. Hint: According to what you just proved, you have to take advantage of the fact that the codomain is not compact. If your function is bounded, you are not taking advantage of this fact. In fact, if your function is bounded, you can restrict its codomain to be compact, and then problem 26.8 implies that the function is continuous. Make sure you prove that the graph is closed. The fact that preimages of closed sets under continuous maps are closed is a useful tool to do so.

(b) The only way compactness enters the proof of problem 26.8 is by letting you apply problem 26.7. Thus, a counterexample to 26.8 should lead to a counterexample to 26.7 when Y is not compact. Find a closed subset of $\mathbb{R} \times \mathbb{R}$ whose projection to the first coordinate is not closed.

Note that since the projection $\pi_1\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous map, you have also found a counterexample to problem 26.6 when X is not compact. Note also that you showed on the second midterm that projections are always open maps, but we see here that they are closed only with additional assumptions.

(10) Do problem 26.12 on page 171. In the special case where $X = Y \times Z$ and $p\colon X \to Y$ is the projection, this statement reduces to our theorem from class that if Y and Z are compact, then so is $Y \times Z$. Consequently, similar proof techniques might be helpful. The hint given in the book corresponds to the tube lemma, for example. You need to prove it, but the proof is much easier because p is assumed to be closed.

This generalization can also be applied to spaces X that are not products. A simple non-product example is the unit circle, $X = S^1 \subseteq \mathbb{R}^2$ mapping to $Y = [-1, 1]$ via projection to one of the coordinates. Here, $p^{-1}(\{y\})$ is usually two points, but $p^{-1}(\{-1\})$ and $p^{-1}(\{1\})$ are one-point sets.

These situations happen frequently in topology and algebraic geometry. We think of $p^{-1}(\{y\})$ as a *fiber* parametrized by $y \in Y$. $p^{-1}(\{y\})$ is usually a nice surface or something, but for some values of y, we get a *singular fiber* where $p^{-1}(\{y\})$ is degenerate in some way. In the circle example above, the generic fiber is a two-point set that varies with y, but when $y = \pm 1$, the two-point set becomes degenerate.

Another nontrivial example to think about is when X is the Möbius strip and Y is its centerline, a circle. We can project each point on the Möbius strip to the corresponding point on the centerline. This example is interesting because, although all of the fibers $p^{-1}(\{y\})$ are homeomorphic to a closed interval $[-1, 1]$, the Möbius strip X is not homeomorphic to $Y \times [-1, 1]$, which would be the cylinder. Thus, the Möbius strip is the simplest example of a nontrivial *fiber bundle*, which is a space that looks like a product space locally, but may or may not be a product space globally.